Lines of curvature and umbilical points for implicit surfaces

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Abstract
This paper develops a method to analyze and compute the lines of curvature and their differential geometry defined on implicit surfaces. With our technique, we can explicitly obtain the analytic formulae of the associated geometric attributes of an implicit surface, e.g., torsion of a line of curvature and Gaussian curvature. Additionally, it can be used to directly derive the closed formulae of principal directions and corresponding principal curvature of an implicit surface. We also present a novel criterion for non-umbilical points and umbilical points on an implicit surface.

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1. Introduction

1.1. Motivation

Principal curvature and principal directions are intrinsic features of smooth surfaces widely used in geometric feature analysis. They play important roles in differential geometry (Do Carmo, 1976; Willmore, 1959), geometric modeling and processing (Hoffmann, 1993; Goldman, 2005; Maekawa et al., 1996) and other applications, such as image and data analysis (Monga and Benayoun, 1995; Sander and Zucker, 1992), shape analysis (Belyaev et al., 1998), surface segmentation (Stylianou and Farin, 2004), remeshing, or non-photorealistic rendering (Alliez et al., 2003).

A line of curvature on a surface is an integral curve of the vector field of principal directions on the surface, whose tangent at each point is a principal direction of the surface. Lines of curvature on parametric surfaces have been a long-standing research focus of differential geometry (Do Carmo, 1976; Willmore, 1959), and also have been researched carefully in geometric modeling (Maekawa et al., 1996).

Parametric and implicit equations are two basic techniques used to produce 3D curved surfaces.

By far, parametric representation dominates most common analytic representations for curves and surfaces in computer-aided geometric modeling, profiting from their convenient analysis and expression of their geometric shape.
and features. For a parametric surface, each surface patch must possess its corresponding 2D parametric domain. It is a non-trivial task to construct proper and global parameterizations for a complex shape. They can not be directly applied to non-parametric surfaces, i.e. implicit surfaces.

In contrast to traditional parametric surfaces, implicit surfaces are independent of parameterizations. They can easily describe smooth and intricate shapes, and possess closure under many geometric operations. Implicit surfaces are finding use in a growing number of geometric modeling and graphics applications (Hoffmann, 1993; Ohtake et al., 2003). Because of the advantages of implicit representation, people are now attaching more and more importance to modeling with implicit surfaces. However, managing complex models remains difficult compared with traditional parametric representation. One main reason is that it is more difficult to capture the geometric features of an implicit surface than those of a parametric one. Because of the convenience of explicit formulae of geometric attributes for computation and analysis, it is valuable to derive the corresponding formulae for implicit surfaces. However, the formulae of geometric attributes for implicit surfaces, such as Gaussian curvature, are not developed so well as those on parametric surfaces, and they are harder to find in the literature (Goldman, 2005).

As one of the significant geometric features, lines of curvature combine most of the important geometric concepts of surfaces, including principal curvature and Gaussian curvature, and are often used in CAGD and computer graphics. However, they are usually derived independently in the literature, and can not be efficiently applied to shape analysis of an implicit surface. For a line of curvature, the most important aspect is its tangent vector field. To the best of our knowledge, none of the related works are analytic approaches to studying the lines of curvature on an implicit surface (see Section 1.2). As a result, they usually agree with numerical computation but not with analysis of geometrical attributes, e.g. torsion and umbilical points. Therefore, we need an efficient method to perform geometric analysis for implicit surfaces while combining those important geometric concepts.

This paper develops such a mathematical method to analyze and compute the lines of curvature and their differential geometry defined on an implicit surface. Our method reduces the geometric analysis involved in lines of curvature for implicit surfaces to a more straightforward and intuitive approach.

1.2. Related works

Generally speaking, the methods of estimation of principal curvature and principal direction can be categorized into three groups by surface representation: a discrete sampling point set, a parametric surface and an implicit surface.

1.2.1. Lines of curvature

(1) Discrete sampling point sets, e.g. a triangular mesh or unorganized set of points, which are approximations of a smooth surface, have become very popular in recent years because of their insensitivity to complex topology of surfaces and the emergence of more powerful display cards. To perform feature analysis, the estimation of curvature hidden behind the point set is often required in advance. This estimation from sampled smooth surfaces typically originates from the discrete approximation of continuous analysis methods of a smooth surface, such as quadric fitting, discrete estimation of the tensor of curvature, and the covariance matrices method; see (Petitjean, 2002) for a survey. However, these methods provide us little help for our work since they are designed for numerical computation of a discrete point set.

(2) Compared with a discrete point set, the research on lines of curvature of parametric surfaces has a long history with sound theoretical foundations, and numerous theoretical results are available. They can not be covered in this paper but some well-known conclusions are listed in the next section of this work. Interested readers can refer to any differential geometry textbook (Do Carmo, 1976; Willmore, 1959). (Maekawa et al., 1996) is also a good overview on lines of curvature defined on a parametric surface, especially on umbilical points.

(3) Mainly two groups of work can be found in the literature relating to lines of curvature defined on an implicit surface. The first group focuses on how to derive associated explicit formulae of geometric attributes, such as principal curvature and Gaussian curvature. The second is closely related to ridges and ravines of an implicit surface, which are based on the study of extrema of the principal curvature along their lines of curvature.

- Goldman (2005) provides a collection of curvature formulae for implicitly defined curves and surfaces, otherwise scattered throughout the geometric modeling literature. However, principal direction and its corresponding normal curvature, i.e. principal curvature, and closely-related lines of curvature are not well covered. Since,
however, Gaussian curvature $K_G$ and mean curvature $K_M$ can be computed explicitly, the two principal curvatures can be formulated as:

$$
\kappa_1 = K_M + \sqrt{K_M^2 - K_G}, \quad \kappa_2 = K_M - \sqrt{K_M^2 - K_G}
$$

Let $n$ be the unit normal vector. Then $\kappa_1$ and $\kappa_2$ are the eigenvalues of $\nabla n$. In order to find their associated principal directions $T_1$ and $T_2$, we only need to compute the associated eigenvectors of $\kappa_1$ and $\kappa_2$, respectively (Belyaev et al., 1998; Turkiyyah et al., 1997; Willmore, 1959):

$$\nabla n \ast T = \kappa T$$

The resulting $T_1$ and $T_2$ are numerical solutions and can not be directly applied to further analysis as analytic variables.

Another method is to search the directions for which the normal curvature is an extremum (Monga and Benayoun, 1995). But this method relies on the choice of the orthonormal basis of the tangent plane at this point, which can not be fixed analytically on the whole surface.

• More closely related to lines of curvature are those in other fields, dealing with feature lines, i.e. ridges and ravines on an implicit surface (Belyaev et al., 1998; Monga and Benayoun, 1995; Ohtake et al., 2004). In these works, they are traced from a starting point on a starting line of curvature to the local extremum on an associated line of curvature.

Interesting work is shown in (Hartmann, 1999), where curvature formulae are derived, via normal forms, and applied to visualization of lines of curvature, feature lines, intersection curves of surfaces, etc. Nevertheless, the mentioned works lack explicit formulae for principal directions. For a line of curvature, its principal directions are more important than its principal curvature, so a key step is to compute the principal directions on a smooth surface, instead of the principal curvature. In this paper we will show that how it will be more convenient for lines of curvature.

1.2.2. Umbilical points

An umbilical point is an important geometric attribute, closely related to lines of curvature. It is a singularity of a line of curvature: a line of curvature will end at such points. In the CAGD area little attention has been paid to umbilical points for detailed shape analysis (Maekawa et al., 1996). It may partly be because there is an effective criterion for a smooth surface defined by a formula, for both parametric or implicit surfaces:

**Lemma 1.1.** A point is an umbilical point if and only if $K_M^2 - K_G = 0$ at this point.

In (Maekawa and Patrikalakis, 1994), a computational method is described to locate all isolated umbilical points on a parametric polynomial surface. For the vector field of a discrete point set, they can be found by going over each point (Alliez et al., 2003; Sander and Zucker, 1992). However, no other criterion for umbilical points on an analytic implicit surface is reported in literature other than Lemma 1.1.

1.3. Our contribution

Lines of curvature and related differential geometry of implicit surfaces are seldom discussed in literature, despite their significance. The aim of this paper is to analyze the line-of-curvature related differential geometry of implicit surfaces. In contrast to other works, our work is based on the analytic formula of principal directions defined on an implicit surface.

Our contributions include:

• Providing an explicit differential equation for lines of curvature defined on an implicit surface. As a result, many properties of differential geometry can be derived from this equation.

• Deriving the explicit formulae of geometric attributes, which are closely associated with lines of curvature.
Proposing a novel criterion for non-umbilical points and umbilical points, more practical than Lemma 1.1, as shown by some examples.

The rest of this paper is organized as follows. In the next section we will give a brief overview of lines of curvature defined on a parametric surface in classical differential geometry. In Section 3, we introduce the differential equation of lines of curvature for an implicit surface, and then transform it into a new one analytically. Based on this equation, we can find the analytic principal directions of each point on the surface. Then in Section 4, we will use these results to derive the associated differential geometry of the lines of curvature: curvature and torsion of lines of curvature, formulae of principal directions and corresponding curvature, and mean and Gaussian curvature. In Section 5, we propose our new criterion for non-umbilical points and umbilical points of an implicit surface, and then a more practical one is proposed in the next section. Illustrative examples and our conclusions are given in the last two sections.

2. Review of differential geometry

In this section, we summarize the relevant definitions and results concerning lines of curvature on a parametric surface.

2.1. Notations

We first introduce some notation and definitions applied in this paper.

In this paper, a vector is a column vector. Bold letters such as $\mathbf{r}$ will be used for vectors and vector functions. Usually, a space curve or a surface is defined by the radius vector $\mathbf{r}(t)$ or $\mathbf{r}(u,v)$ as a parametric representation. An implicit surface is defined as the locus of points whose coordinates $(x,y,z)^T$ are restricted within $H(x,y,z) = 0$. We assume that they are smooth enough so that they possess a sufficient number of continuous (partial) derivatives supplied in the paper. The triple scalar product of three vectors $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$, $\mathbf{t}_1 \times \mathbf{t}_2 \cdot \mathbf{t}_3$, is denoted by $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$.

$s$ denotes the arc length of a curve $\mathbf{r}(t)$, and for the sake of simplicity, we will also use the notation $d\mathbf{r}/ds = \dot{\mathbf{r}}$ and $d^2\mathbf{r}/ds^2 = \ddot{\mathbf{r}}$.

In the three-dimensional space, a scalar-valued function $H(x,y,z)$ has three partial derivatives. The gradient of $H(x,y,z)$ is the vector of partial derivatives:

$$\nabla H = \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

(2)

The gradient of a vector field $\nabla H$ is found by applying the gradient operator to each of the components of the vector field:

$$\nabla^2 H = \begin{pmatrix} H_{xx} & H_{xy} & H_{xz} \\ H_{yx} & H_{yy} & H_{yz} \\ H_{zx} & H_{zy} & H_{zz} \end{pmatrix}$$

(3)

Eq. (3) is a symmetric matrix if $H$ is at least $C^2$.

The matrix multiplication of two matrices $M_1$ and $M_2$ is written as $M_1 \ast M_2$; the scalar product of two vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ is denoted as $\mathbf{t}_1 \cdot \mathbf{t}_2$.

2.2. Lines of curvature and umbilical points

The first and second fundamental forms of a surface, denoted by $I$ and $II$ respectively, are important geometric structures defined by the surface (Do Carmo, 1976; Willmore, 1959). The normal curvature $\kappa_n$ of $\mathbf{r}(u,v)$ at $p$ in the direction $\mathbf{T}$ is the curvature of the plane curve at this point formed by the intersection of the plane defined by $\mathbf{T}$ and $\mathbf{n}$ with the surface. Additionally, we know that $\kappa_n = \frac{II}{I}$. $\kappa_n$ depends on $\mathbf{T}$ and that the directions for which $\kappa_n$ takes maximum and minimum values are called the principal directions. The corresponding normal curvatures are the principal curvatures, which represent the maximum and minimum $\kappa_n$ values, as in Eq. (1). If $\kappa_1 = \kappa_2$, however,
the normal curvature in all directions is the same; this point is called an umbilical point, each direction of which is a principal direction.

We have an important theorem about umbilical points:

**Theorem 2.1.** If all points of a continuous surface $S$ are umbilical points, then $S$ is either part of a plane or part of a sphere.

Thus an umbilical point is either a planar point or a spherical point.

Usually, a point $p$ on a smooth surface has either only two principal directions, or an infinite number of principal directions with the same normal curvature. Therefore,

**Theorem 2.2.** $p$ is a non-umbilical point if and only if there exist only two orthogonal principal directions at $p$.

A line of curvature is a curve on a surface whose tangents are the principal directions at all of its points. The lines of curvature have been well studied in the classical theory of differential geometry defined on a parametric surface (Do Carmo, 1976; Willmore, 1959).

### 3. Lines of curvature on an implicit surface

In this section, we will expand on the lines of curvature on an implicit surface.

Given an implicit surface $H(x, y, z) = 0$, we assume it is regular, i.e. $\nabla H \neq 0$. For the sake of discussion, only $H(x, y, z) = 0$ without umbilical points is under consideration in this section.

**3.1. Differential equation of lines of curvature**

First, we describe a known theorem, which characterizes lines of curvature on a surface, as shown by Monge:

**Theorem 3.1.** A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

The proof for Theorem 3.1 can be found in text books on differential geometry, such as (Willmore, 1959). Based on this, we can directly derive the differential equation of lines of curvature for an implicit surface.

Suppose $\mathbf{r}(s) = (x(s), y(s), z(s))^T$ is a curve lying on $H(x, y, z) = 0$, and $\mathbf{n}$ denotes the unit normal of $H(x, y, z) = 0$. According to Theorem 3.1, $\mathbf{r}(s)$ is a line of curvature if and only if

$$ (\dot{\mathbf{r}}, \mathbf{n}, \dot{\mathbf{n}}) = 0 $$

(4)

Because $\mathbf{n} = \begin{pmatrix} \frac{\nabla H}{|\nabla H|} \end{pmatrix}$, by differentiating both sides we obtain

$$ d\mathbf{n} = \frac{1}{|\nabla H|} \left( d\nabla H - d|\nabla H| \mathbf{n} \right) $$

(5)

Substituting Eq. (4) by Eq. (5), we have

$$ (\dot{\mathbf{r}}, \mathbf{n}, \dot{\mathbf{n}}) = \frac{1}{|\nabla H|} \left( (\dot{\mathbf{r}}, \mathbf{n}, (\nabla H)^{\top}) - (\nabla H) \dot{\mathbf{n}} \right) = \frac{1}{|\nabla H|^2} (\dot{\mathbf{r}}, \nabla H, (\nabla H)^{\top}) $$

(6)

We obtain the differential equation of the lines of curvature:

$$ (\dot{\mathbf{r}}, \nabla H, (\nabla H)^{\top}) = 0 $$

(7)

which may be written, in a more symmetric way, as:

$$ I \begin{pmatrix} dx & dy & dz \\ H_x & H_y & H_z \\ dH_x & dH_y & dH_z \end{pmatrix} = 0 $$

(8)

Eq. (8) is the differential equation of lines of curvature for $H(x, y, z) = 0$.

Following from Theorem 3.1, we find
Corollary 3.2. \( dr = (dx, dy, dz)^T \) at a point \( p \) of \( H(x, y, z) = 0 \) is a principal direction if and only if \( I(dr) = 0 \).

Eq. (8) is difficult to use in practice. We need to derive a useful equivalent described by an explicit formula of principal directions.

3.2. Principal directions for an implicit surface

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature. A line of curvature on \( H(x, y, z) = 0 \) is a curve integral along principal directions on the vector field, so we first must compute the principal directions for each point of \( H(x, y, z) = 0 \).

We note that \( I(dx, dy, dz) \) can be expanded to be a quadratic equation with respect to \( dx, dy \) and \( dz \):

\[
I(dx, dy, dz) = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}
\]

\[
= Adx^2 + 2B dx dy + 2C dx dz + D dy^2 + 2E dy dz + F dz^2
\]

where \( A, \ldots, F \) are functions of polynomial combinations of the first derivatives and second derivatives of \( H(x, y, z) \) with respect to \( x, y \) and \( z \), respectively:

\[
A = H_y H_z x - H_z H_y x \\
B = (H_z H_{xx} - H_x H_{zx} + H_y H_{zy} - H_z H_{yy})/2 \\
C = (H_y H_{zz} - H_z H_{yz} + H_x H_{yx} - H_y H_{xx})/2 \\
D = H_z H_{xy} - H_x H_{yz} \\
E = (H_x H_{yy} - H_y H_{xy} + H_z H_{xz} - H_x H_{zz})/2 \\
F = H_x H_{yz} - H_y H_{xz}
\]

On the other hand, since \( dr = (dx, dy, dz)^T \) is the tangent vector on a point of a line of curvature \( r \) on \( H(x, y, z) = 0 \), it is perpendicular to the normal at this point, i.e.

\[
II(dr) = \nabla H \cdot dr = H_x dx + H_y dy + H_z dz = 0
\]

Thus, \( dr = (dx, dy, dz)^T \) satisfies the following equations:

\[
\begin{align*}
I(dr) &= 0 \\
II(dr) &= 0
\end{align*}
\]

Since \( H(x, y, z) = 0 \) is a regular surface, \( \nabla H \neq 0 \). Without losing generality, let \( H_z \neq 0 \), and substituting \( dz \) from \( II(dr) = 0 \) to \( I(dr) = 0 \) in Eq. (11), we have

\[
III(dx, dy) = (AH_z^2 - 2CH_y H_z + FH_y^2) dx^2 + 2(BH_z^2 - CH_y H_z + EH_z H_x + FH_x H_y) dx dy \\
+ (DH_z^2 - 2EH_z H_y + FH_y^2) dy^2 = 0
\]

For the sake of description, we write the coefficients in Eq. (12) as \( U, V, \) and \( W \):

\[
U = AH_z^2 - 2CH_y H_z + FH_y^2 \\
V = 2(BH_z^2 - CH_y H_z - EH_z H_x + FH_x H_y) \\
W = DH_z^2 - 2EH_z H_y + FH_y^2
\]

\( III(dx, dy) = 0 \) is a quadratic equation with respect to \( dx \) and \( dy \), and its discriminant is \( \Delta = V^2 - 4UW \), with regard to which we have a theorem:

3 We obtain Eq. (12) by assuming \( H_z \neq 0 \); if not, alternate formulae may be found by cyclic permutation of \( x, y \) and \( z \).
Theorem 3.3. Given a point \( p = (x, y, z)^T \) of \( H(x, y, z) = 0 \):

1. \( p \) is a non-umbilical point if and only if \( \Delta > 0 \).
2. \( p \) is an umbilical point if and only if \( \Delta = 0 \).

Theorem 3.3 tells us that there are exactly two solutions of Eq. (12), corresponding to each of the principal directions, at points which are not umbilical. Its proof will be given in Section 5.

A corollary follows from Theorem 3.3:

Corollary 3.4. \( \Delta \geq 0 \).

Now we prepare to compute the solutions of Eq. (11):

1. \( U \neq 0 \) or \( W \neq 0 \)
   Suppose \( U \neq 0 \). The solutions are:
   \[
   dx : dy = c_1 = \left( -V + \sqrt{V^2 - 4UW} \right) / 2U
   \]
   and
   \[
   dx : dy = c_2 = \left( -V - \sqrt{V^2 - 4UW} \right) / 2U
   \]
   Thus
   \[
   dx : dy : dz = c_i H_z : H_z : (-c_i H_x - H_y), \quad i = 1, 2.
   \]
   Without a loss of generality, we will only take \( dx : dy = c_1 \) into account for derivation. Therefore,
   \[
   dx : dy : dz = c_1 H_z : H_z : (-c_1 H_x - H_y) = \left( -V + \sqrt{V^2 - 4UW} \right) H_z : 2U H_z : \left( V - \sqrt{V^2 - 4UW} \right) H_x - 2U H_y
   \]
   So the tangent vector of the line of curvature is:
   \[
   T_1 = \left( \frac{-V + \sqrt{V^2 - 4UW} H_z}{2U H_x}, \frac{V - \sqrt{V^2 - 4UW} H_x - 2U H_y}{2U H_z} \right)
   \]
   The other one line of curvature for \( c_2 \) can be computed in much the same way as:
   \[
   T_2 = \left( \frac{-V - \sqrt{V^2 - 4UW} H_z}{2U H_x}, \frac{V + \sqrt{V^2 - 4UW} H_x - 2U H_y}{2U H_z} \right)
   \]
2. \( U = 0 \) and \( W = 0 \)
   Thus \( dx = 0 \) or \( dy = 0 \).
   If \( dx = 0 \), then
   \[
   T_1 = (0, H_z, -H_y)^T
   \]
   If \( dy = 0 \), we obtain:
   \[
   T_2 = (H_z, 0, -H_x)^T
   \]

We write \( T_i = (X, Y, Z)^T, i = 1, 2 \). \( T_i \) is the principal direction. Thus the differential equation of a line of curvature on \( H(x, y, z) = 0 \) can be rewritten as
\[
\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}
\]
Given a starting point \( p_0 \) and an initial principal direction \( t_0 \), Eq. (19) defines one and only one line of curvature on \( H(x, y, z) = 0 \).
4. Differential geometry of lines of curvature

In this section, we will derive the differential geometry properties associated with lines of curvature on \( H(x, y, z) = 0 \), and related geometric parameters, entirely based on Eq. (19).

4.1. Orthogonality

Since \( T_1 \) and \( T_2 \) are the only two principal directions, they should be orthogonal. We will now verify this assumption.

By computation, we find that:

\[
(H_y^2 + H_z^2)U - (H_x H_y)V + (H_x^2 + H_z^2)W = 0
\]  
(20)

Therefore, if \( U \neq 0 \), then

\[
T_1 \cdot T_2 = \frac{(H_y^2 + H_z^2)U - (H_x H_y)V + (H_x^2 + H_z^2)W}{U} = 0
\]

It can also be verified that \( T_1 \cdot T_2 = 0 \) holds for \( W \neq 0 \).

If \( U = W = 0 \), then \( V \neq 0 \) for non-umbilical points according to Theorem 3.3, and \( H_x H_y = 0 \) follows from Eq. (20). Thus

\[
T_1 \cdot T_2 = H_x H_y = 0
\]

To sum up, we always have

\[
T_1 \cdot T_2 = 0
\]  
(21)

4.2. Differential geometry associated with lines of curvature

We will call the principal direction field \( T \), standing for \( T_1 \) or \( T_2 \). Another way to define lines of curvature in a differential equation can then be:

\[
\dot{r} = \frac{T}{|T|}
\]  
(22)

Eq. (22) can be applied to the numerical computation of lines of curvature and derivation of geometric parameters such as curvature.

For a space curve, the most important measures are its curvature \( \kappa \) and torsion \( \tau \). In fact, the shape of a space curve is fixed when its curvature and torsion are fixed. The curvature \( \kappa \) and the torsion \( \tau \) can be computed by Eq. (22) as the following formulae (Goldman, 2005):

\[
\kappa = \frac{|T \times (\nabla T \ast T)|}{|T|^3}
\]  
(23)

and

\[
\tau = \frac{(T, \nabla T \ast T, T^T \ast \nabla^2 T \ast T + (\nabla T)^2 \ast T)}{|T \times (\nabla T \ast T)|^2}
\]  
(24)

Note that \( \dot{T} = \frac{\nabla T \ast T}{|T|^2} \). For each point of a line of curvature of \( H(x, y, z) = 0 \), the Frenet frame can be computed as:

\[
\begin{align*}
\alpha &= \frac{T}{|T|} \\
\beta &= \gamma \times \alpha \\
\gamma &= \frac{T \times (\nabla T \ast T)}{|T \times (\nabla T \ast T)|}
\end{align*}
\]  
(25)

where \( \alpha \) is the unit tangent, \( \beta \) is the unit principal normal, and \( \gamma \) is the unit binormal.

Thus,
\[ \beta = \gamma \times \alpha = \frac{T \times (\nabla T^* \times T)}{|T||T \times (\nabla T^* T)|} = \frac{(T \cdot T)(\nabla T^* T) - (T \cdot (\nabla T^* T))T}{|T||T \times (\nabla T^* T)|} \]

\[ = \left( \frac{T \cdot (\nabla T^* T)}{|T||(\nabla T^* T) \times T|} \right) \times \left( \frac{|T|}{|T|} \right) \times (\nabla T^* T) \times T \]

(26)

In the principal direction \( T_1 \) or \( T_2 \), its normal curvature is exactly the principal curvature \( \kappa_i \) of \( T_i \), \( i = 1, 2 \). Thus:

\[ \kappa_i = (n \cdot \beta)\kappa = \frac{n \cdot (\nabla T_i^* T_i)}{|T_i|^2} = \frac{\nabla H \cdot (\nabla T_i^* T_i)}{|\nabla H||T_i|^2} \]

Therefore, we arrive at the Gaussian curvature \( K_G \) formula,

\[ K_G = \kappa_1\kappa_2 = \frac{(\nabla H \cdot (\nabla T_1^* T_1)) \cdot (\nabla H \cdot (\nabla T_2^* T_2))}{|\nabla H|^2|T_1|^2|T_2|^2} \]

(27)

and the mean curvature \( K_M \) formula,

\[ K_M = \frac{\kappa_1 + \kappa_2}{2} = \frac{\nabla H}{2|\nabla H|} \cdot \left( \frac{\nabla T_1^* T_1}{|T_1|^2} + \frac{\nabla T_2^* T_2}{|T_2|^2} \right) \]

(28)

Furthermore, we can also easily arrive at the normal curvature \( \kappa_n \) in any tangent direction \( \tilde{T} \) by the Euler Formula:

\[ \kappa_n = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi \]

where \( \varphi \) is the angle between \( T_1 \) and \( \tilde{T} \).

5. Criterion of umbilical points

The aforementioned can only hold true for non-umbilical points. At umbilical points, the normal curvatures in all directions are equal, and the principal directions are indeterminate. Therefore the orthogonal set of lines of curvature may have singular properties at an umbilical point. In this section, we will propose a novel sufficient and necessary condition to distinguish them.

We assume that \( H_z \neq 0 \).

We already know that there are no less than two principal directions for each point of \( H(x, y, z) = 0 \). Because Eq. (8) is equivalent to Eq. (11), we have

Lemma 5.1.

\[
\begin{align*}
I &= 0 \\
II &= 0
\end{align*}
\]

has no less than two solutions.

So there is always a non-zero vector \( dr = (dx, dy, dz)^T \) satisfying Eq. (8), and if \( dr \) is a solution, so are all \( k dr \), where \( k \) is any real number. We observe that \( 0 \) always satisfies Eq. (8), but we regard it as a degenerative case of \( k dr \) with \( k = 0 \). Thus in this paper, a solution of Eq. (8) indicates a non-zero value.

On the other hand, since \( dr \) is parallel to \( k dr \), they actually correspond to the same principal direction. Therefore, \( dr \) and \( k dr \) are the same solution. Furthermore, \( dr_1 \) and \( dr_2 \) are distinct if and only if \( dr_1 \) is not parallel to \( dr_2 \). Thus

Lemma 5.2. \( dr \) is a principal direction if and only if \( dr \) is a solution of

\[
\begin{align*}
I &= 0 \\
II &= 0
\end{align*}
\]

With Eq. (12), we have

Lemma 5.3. If \( p \) is a non-umbilical point, \( III = 0 \) has two distinct solutions.
Proof. \( p \) is a non-umbilical point, so there are two principal directions \( \mathbf{d}r_1 = (dx_1, dy_1, dz_1)^T \) and \( \mathbf{d}r_2 = (dx_2, dy_2, dz_2)^T \) satisfying Eq. (11). Because \( \mathbf{d}r_1 \) and \( \mathbf{d}r_2 \) are orthogonal, either \( (dx_1, dy_1)^T \neq 0 \) or \( (dx_2, dy_2)^T \neq 0 \). Without a loss of generality, let \( (dx_1, dy_1)^T \neq 0 \).

If \( (dx_2, dy_2)^T = k(dx_1, dy_1)^T \), then

\[
\mathbf{n} = \frac{\mathbf{d}r_1 \times \mathbf{d}r_2}{|\mathbf{d}r_1 \times \mathbf{d}r_2|} = \frac{1}{|\mathbf{d}r_1 \times \mathbf{d}r_2|} \begin{pmatrix}
    dy_1 & dz_1 & dx_1 \\
    dy_2 & dz_2 & dx_2 \\
    0 & 0 & 0
\end{pmatrix}^T
\]

Eq. (31) is in contradiction to the assumption \( H_z \neq 0 \). So \( (dx_1, dy_1)^T \) is not parallel to \( (dx_2, dy_2)^T \), and they are two distinct solutions of Eq. (12). \( \square \)

The following theorem shows the relationship between Eq. (11) (or Eq. (12)) and an umbilical point

Theorem 5.4. (1) \( p \) is a non-umbilical point if and only if \( III = 0 \) has only two solutions.

(2) \( p \) is a non-umbilical point if and only if

\[
\begin{cases}
    I = 0 \\
    II = 0
\end{cases}
\]

has only two solutions.

Proof. We only prove (1) since (2) follows from (1).

If \( p \) is a non-umbilical point, the assertion follows from Lemma 5.3. We now assume that Eq. (12) has only two solutions \( (dx_1, dy_1)^T \) and \( (dx_2, dy_2)^T \). From \( II = 0 \) of Eq. (11), we get \( \mathbf{d}r_1 = (dx_1, dy_1, -\frac{H_z dx_2 + H_x dy_2}{H_z})^T \) and \( \mathbf{d}r_2 = (dx_2, dy_2, -\frac{H_z dx_2 + H_x dy_2}{H_z})^T \). Obviously, \( \mathbf{d}r_1 \) and \( \mathbf{d}r_2 \) are only two solutions of Eq. (11). By Lemma 5.2, \( \mathbf{d}r_1 \) and \( \mathbf{d}r_2 \) are two principal directions. According to Eq. (21), we also obtain that \( \mathbf{d}r_1 \cdot \mathbf{d}r_2 = 0 \). Therefore, \( H(x, y, z) = 0 \) has only two orthogonal principal directions \( \mathbf{d}r_1 \) and \( \mathbf{d}r_2 \), resulting in that \( p \) is a non-umbilical point by Theorem 2.2. \( \square \)

Now we will discuss the remaining problem from Section 3.1: the proof of Theorem 3.3. Although the first assertion can follow from Theorem 5.4 directly, we propose a new proof with highly a geometric meaning.

The quadratic form \( I(dx, dy, dz) \) is totally determined by its matrix of coefficients:

\[
M = \begin{pmatrix}
    A & B & C \\
    B & D & E \\
    C & E & F
\end{pmatrix}
\]

We can analyze the criterion of non-umbilical points by \( M \) according to the classification criterion of analytic geometry about a quadric surface.

(1) \( M = 0 \), i.e. all of \( A, B, \ldots, F \) are zeros

Then \( I(dx, dy, dz) = 0 \) is an identical equation and \( \Delta = 0 \). The solution space is the tangent plane \( II(dx, dy, dz) = 0 \). \( p \) can not be a non-umbilical point.

(2) \( M \neq 0 \), i.e. at least one of \( A, B, \ldots, F \) is non-zero

According to the classification of \( I(dx, dy, dz) = 0 \) in Appendix A, we have

(a) If \( I_3 \neq 0 \), \( I(dx, dy, dz) = 0 \) is a real quadric cone. Then the plane \( II(dx, dy, dz) = 0 \) intersects \( I(dx, dy, dz) = 0 \) at two generatrices of \( I(dx, dy, dz) = 0 \), as shown in Fig. 1(a), if and only if \( p \) is a non-umbilical point. \( III(dx, dy) = 0 \) is the projection of the two generatrices on the \( dx-\) plane. By Lemma 5.3, it is two distinct lines intersecting at the origin in the \( dx-\) plane. So that \( p \) is a non-umbilical point if and only \( \Delta > 0 \).

(b) If \( I_3 = 0 \), \( I(dx, dy, dz) = 0 \) is a pair of real intersecting planes, passing through the origin. Then \( p \) is a non-umbilical point if and only if neither of them coincides with \( II(dx, dy, dz) = 0 \). Then the plane
II(dx, dy, dz) = 0 must intersect I(dx, dy, dz) = 0 at two lines of I(dx, dy, dz) = 0, as shown in Fig. 1(b). As in the case I_3 \neq 0, \Delta > 0 should still hold. If II = 0 coincides with one plane of I = 0, \( p \) is an umbilical point.

Therefore, \( p \) being a non-umbilical point is equivalent to the discriminant of III(dx, dy) = 0 being larger than zero (from Theorem 5.4), resulting in the first assertion of Theorem 3.3.

To prove the second assertion, we first suppose \( \Omega \) is the maximum connected set of umbilical points of \( H(x, y, z) = 0 \). We denote the closure of \( \Omega \) by \( \bar{\Omega} \) whose boundary is \( \Omega_1 \) and interior is \( \Omega_2 \). It suffices to show that \( \Delta(\Omega_1) = 0 \) and \( \Delta(\Omega_2) = 0 \):

1. Due to the continuity of \( \Delta \), it must be true that \( \Delta(\Omega_1) = 0 \), according to (1) of Theorem 3.3.
2. If \( \Omega_2 \neq \emptyset \), then according to Theorem 2.1, \( H(x, y, z) = 0 \) defined on \( \Omega_2 \) is either contained in a plane or in a sphere. It can be easily verified that \( \Delta(\Omega_2) = 0 \) for a plane or a sphere.

Therefore \( \Delta(\Omega) = \Delta(\bar{\Omega}) = 0 \). Here, we can easily observe that \( \Omega = \bar{\Omega} \).

Thus we have proven Theorem 3.3.

6. Further discussion

In this section, we will give an equivalent of Theorem 3.3, but more convenient in practice.

For the sake of clarity, we summarize the reasoning of Section 5 in Table 1.

According to Table 1, \( p \) is an umbilical point if and only if the solution space of Eq. (11) is the whole II(dx, dy, dz) = 0. If \( H_z \neq 0 \), its projection onto the dx–dy plane covers the coordinate plane in whole. So

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Classification of solution space ( {I = 0} \cap {II = 0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M = 0 )</td>
<td>( M \neq 0 )</td>
</tr>
<tr>
<td>I = 0</td>
<td>3D space</td>
</tr>
<tr>
<td>Solution space</td>
<td>II = 0</td>
</tr>
<tr>
<td>III = 0</td>
<td>dx–dy plane</td>
</tr>
<tr>
<td>Conditions</td>
<td>( III \equiv 0 )</td>
</tr>
</tbody>
</table>
III(dx, dy) = 0 must be an identical equation, i.e. U = V = W = 0; if Δ > 0, p is a non-umbilical point, resulting in that at least one of U, V, and W are non-zero.

Therefore, we have

**Theorem 6.1.** Given a point \( p = (x, y, z)^T \) of \( H(x, y, z) = 0 \):

1. \( p \) is an umbilical point if and only if \( U = V = W = 0 \).
2. \( p \) is a non-umbilical point if and only if one of \( U, V \) and \( W \) is non-zero.

**Remark 6.2.** We should point out that Eq. (12) is derived based on the assumption that \( H_z \neq 0 \). Roughly speaking, the above conclusions hold true, but III needs to be computed according to the non-zero component of \( \nabla H \), i.e. \( H_x \) or \( H_y \), if \( H_z = 0 \).

7. **Illustrative examples**

For illustrative purposes we present applications of our method as examples.

**Example 1.** The implicit surface is a hyperbolic paraboloid \( \frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z = 0 \). Here we can compute

\[
\begin{align*}
\text{I}(dx, dy, dz) &= \frac{-4}{a^2b^2}[(a^2 + b^2) dx dy - y dx dz + x dy dz] = 0 \\
\text{II}(dx, dy, dz) &= 2\left(\frac{1}{a^2} dx - \frac{y}{b^2} dy - dz\right) = 0
\end{align*}
\]

Eliminating \( dz \) in Eq. (33), we have

\[
\frac{xy}{a^2} dx^2 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + a^2 + b^2\right) dx dy + \frac{xy}{b^2} dy^2 = 0
\]

Now, from Eq. (34), \( U = \frac{xy}{a^2}, V = -(\frac{x^2}{a^2} + \frac{y^2}{b^2} + a^2 + b^2) \), and \( W = \frac{xy}{b^2} \). We can easily obtain the lines of curvature as:

\[
\frac{dx}{-V + \sqrt{V^2 - 4x^2y^2/a^2b^2}} = \frac{dy}{2xy/a^2} = \frac{dz}{(\sqrt{V^2 - 4x^2y^2/a^2b^2} - 2xy^2/a^2b^2)}
\]

or

\[
\frac{dx}{-V - \sqrt{V^2 - 4x^2y^2/a^2b^2}} = \frac{dy}{2xy/a^2} = \frac{dz}{(\sqrt{V^2 - 4x^2y^2/a^2b^2} + 2xy^2/a^2b^2)}
\]

Eqs. (35) and (36) can be applied to the computation of lines of curvature and the associated geometric attributes. Because \( \Delta = V^2 - 4UW = (\frac{x^2}{a^2} - \frac{y^2}{b^2})^2 + 2(\frac{x^2}{a^2} + \frac{y^2}{b^2})(a^2 + b^2) + (a^2 + b^2)^2 > 0 \), there are no umbilical points on a hyperbolic paraboloid.

**Example 2.** An ellipsoid \( H(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \), where \( a > b > c > 0 \).

It is well known that there are four umbilical points on such an ellipsoid. Now we will confirm this with our proposed technique.

By computation and by assuming \( H_z \neq 0 \), we obtain

\[
\begin{align*}
\text{III}(dx, dy) &= -\frac{x}{a^2b^2c^2}(\frac{1}{c^2} - \frac{1}{a^2}) dx^2 \\
&+ \frac{z}{c^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\left(\frac{z}{c^2}\right)^2 - \left(\frac{1}{b^2} - \frac{1}{c^2}\right)\left(\frac{x}{a^2}\right)^2 - \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\left(\frac{y}{b^2}\right)^2 dx dy \\
&- \frac{x}{a^2b^2c^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) dy^2 = 0
\end{align*}
\]
According to Theorem 6.1, for $H(x, y, z) = 0$ with $H_z \neq 0$, i.e. $z \neq 0$, we compute four umbilical points:

$$
\left( \pm a \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, 0, \pm b \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \right)^T
$$

(37)

For those points $H_z = 0$, i.e. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, we apply $H_x \neq 0$, and we get the same umbilical points as Eq. (37). However, they are not covered in the searching domain of $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, so they can be rejected. Finally, two points $(0, \pm b, 0)$ remain to be checked, which can not be verified by $H_z \neq 0$ or $H_x \neq 0$. So we use $H_y \neq 0$. Similarly, no more umbilical points than Eq. (37) can be found.

**Example 3.** A rounded octahedron $H(x, y, z) = x^4 + y^4 + z^4 + 6x^2y^2 + 6y^2z^2 + 6z^2x^2 - 1 = 0$.

$U$, $V$ and $W$ can be computed as:

$U = xyz(z^2 - x^2)f(x^2, y^2, z^2)$

$V = -z(x^2 - y^2)g(x^2, y^2, z^2)$

$W = xyz(y^2 - z^2)h(x^2, y^2, z^2)$

where $f$, $g$, and $h$ are polynomials, each term consisting of a positive constant multiplier and integral powers with respect to $x^2$, $y^2$ and $z^2$ as below:

$$
f = 1536(5x^6 + 31x^4y^2 + 27z^2x^2 + 106y^2z^2x^2 + 51x^2y^4 + 27z^4x^2 + 31z^4y^2 + 5z^6 + 9y^6 + 51y^4z^2)
$$

$$
g = 3072(6x^6y^2 + 9x^6z^2 + 33z^4x^4 + 20x^4y^4 + 51z^2x^4y^2 + 19z^6x^2 + 84z^4x^2y^2 + 6x^2y^6 + 51y^4z^2x^2
+ 9z^2y^6 + 33z^4y^4 + 3z^8 + 19z^6y^2)
$$

$$
h = 1536(5y^6 + 31y^4x^2 + 27y^4z^2 + 106y^2z^2x^2 + 51x^4y^2 + 27z^4y^2 + 31z^4x^2 + 5z^6 + 9x^6 + 51z^2x^4)
$$

We note that $x^2 + y^2 + z^2 \neq 0$ for $(x, y, z)$ on $H(x, y, z) = 0$; so $f > 0$, $g > 0$, and $h > 0$. By $H_z \neq 0$, we have $z \neq 0$. Thus we can derive that $|x| = |y| = |z|$ or $x = y = 0$ for umbilical points by Theorem 6.1. Therefore the umbilical points in the case of $H_z \neq 0$ are

$$
\left( \pm \frac{1}{\sqrt{21}}, \pm \frac{1}{\sqrt{21}}, \pm \frac{1}{\sqrt{21}} \right)^T, \quad (0, 0, \pm 1)^T
$$

(38)

In much the same way as Example 2, we obtain the remaining umbilical points for $H_y \neq 0$ and $H_z \neq 0$ respectively as:

$(0, \pm 1, 0)^T$ (39)

and

$(\pm 1, 0, 0)^T$ (40)

Thus there are 14 umbilical points on this rounded octahedron listed in Eqs. (38)–(40), respectively.

8. Conclusion and future work

In this paper, we have made a careful mathematical study of lines of curvature defined on implicit surfaces. Not only has the differential equation been derived, but many explicit analytic formulae of differential geometry related to lines of curvature have been as well. We have also proposed a novel and practical criterion for umbilical points, whose geometric meaning is illuminated intuitively based on the classification theory of quadric surfaces.

The proposed method is entirely designed from the standpoint of implicit surfaces. The main advantage of our method is that it is based on the explicit formulae of the geometric attributes. It can adapt well to the shape analysis of implicit surfaces, but such a study is beyond the scope of this paper. The numerical implementation of our method presented here is also of importance. These issues will be discussed in future work.
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Appendix A. Classification of quadric surfaces

A quadratic equation is defined as:

\[ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \]  

(A.1)

where \( a^2 + b^2 + c^2 + f^2 + g^2 + h^2 \neq 0 \).

Given an equation of the kind in Eq. (A.1), we can represent one of seventeen different kinds of surfaces, called quadric surfaces.

We now define six functions with respect to Eq. (A.1):

\[ I_1 = a + b + c \]  

(A.2)

\[ I_2 = \begin{vmatrix} b & f \\ f & c \end{vmatrix} + \begin{vmatrix} c & g \\ c & a \end{vmatrix} + \begin{vmatrix} a & h \\ h & b \end{vmatrix} \]  

(A.3)

\[ I_3 = \begin{vmatrix} a & h \\ h & b \end{vmatrix} \begin{vmatrix} f & c \\ g & f \end{vmatrix} \]  

(A.4)

\[ I_4 = K_3 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \]  

(A.5)

\[ K_2 = \begin{vmatrix} b & f & v \\ f & c & w \\ v & w & d \end{vmatrix} + \begin{vmatrix} c & g & w \\ g & a & u \\ w & u & d \end{vmatrix} + \begin{vmatrix} a & h & u \\ h & b & v \\ u & v & d \end{vmatrix} \]  

(A.6)

\[ K_1 = \begin{vmatrix} a & u \\ b & v \\ c & w \\ d \end{vmatrix} \]  

(A.7)

We now apply Eqs. (A.2)–(A.7) to Eq. (9) to classify them:

\[ I_1 = A + D + F = 0 \]  

(A.8)

\[ I_2 = \begin{vmatrix} D & E \\ E & F \end{vmatrix} + \begin{vmatrix} F & C \\ D & A \end{vmatrix} + \begin{vmatrix} A & B \\ B & D \end{vmatrix} \]

\[ = (DF + FA + AD) - (E^2 + C^2 + B^2) \]

\[ = -\frac{1}{2}(A^2 + D^2 + F^2 + 2(E^2 + C^2 + B^2)) < 0 \]  

(A.9)

\[ I_3 = \begin{vmatrix} A & B & C \\ B & D & E \\ C & E & F \end{vmatrix} \]  

(A.10)

Because there are no linear and absolute terms in \( I(dx, dy, dz) \), we know that

\[ I_4 = K_3 = K_2 = K_1 = 0 \]  

(A.11)

If \( I_3 \neq 0 \), \( I(dx, dy, dz) = 0 \) is a pair of real quadric cones for \( I_2 < 0 \) and \( K_3 = 0 \), whose vertices coincide with the origin; if \( I_3 = 0 \), \( I(dx, dy, dz) = 0 \) is a pair of intersecting planes, also passing through the origin.

The detailed classification of quadric surfaces by Eqs. (A.2)–(A.7), can be found in textbooks on Analytic Geometry, such as (Sommerville, 1934).
References


