Noise Properties of Chord-Image Reconstruction

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Abstract—Recently, there has been much progress in algorithm development for image reconstruction in cone-beam computed tomography (CT). Current algorithms, including the chord-based algorithms, now accept minimal data sets for obtaining images on volume regions-of-interest (ROIs) thereby potentially allowing for reduction of X-ray dose in diagnostic CT. As these developments are relatively new, little effort has been directed at investigating the response of the resulting algorithm implementations to physical factors such as data noise. In this paper, we perform an investigation on the noise properties of ROI images reconstructed by using chord-based algorithms for different scanning configurations. We find that, for the cases under study, the chord-based algorithms yield images with comparable quality. Additionally, it is observed that, in many situations, large data sets contain extraneous data that may not reduce the ROI-image variances.

Index Terms—Chord, computed tomography (CT), cone-beam CT, noise, reconstruction.

I. INTRODUCTION

IN recent years, exact algorithms have been developed for reconstructing images [1] and for reconstructing images on “ π -lines” [2]–[4] from helical cone-beam data. Since 2005, papers have been published on algorithm development for reconstructing images on chords for general trajectories [5]–[8]. Some of these algorithms can reconstruct images within 3-D regions of interest (ROIs) from cone-beam data containing both longitudinal and transverse truncations. The introduction of the M -line concept and reconstruction [5], [9] provides additional flexibility for covering volume ROIs.

As these algorithm developments are relatively recent, little effort has been directed at investigating their noise properties. With the algorithm development for ROI-image reconstruction, it has been tacitly assumed that the reduction in necessary scanning angle and in projection data may lead to ROI images from less radiation exposure. This conclusion may, however, depend

on the noise properties of reconstruction algorithms. If ROI reconstruction from the minimal (or reduced) data set leads to noisier ROI images than reconstruction of the same ROI from a larger data set, it may be necessary to increase the X-ray source intensity for the ROI data set to attain the same image quality as those reconstructed from larger data sets. Such an increase can offset the fact that reduced or minimum projection data are needed for ROI reconstruction.

The focus of this paper is to investigate the noise properties of image reconstruction from minimal data set and large data set by use of chord-based algorithms. We demonstrate that the minimal data set can indeed lead to actual reduction of radiation exposure for attaining comparable image quality, defined in terms of image variance, as that obtained with a larger data set. In Section II, we briefly summarize the chord-based reconstruction algorithms: backprojection filtration (BPF) [2], [6], minimum data filtered backprojection (MDFBP) [4], [6], and filtered backprojection (FBP) [6], [10] algorithms. In Section III, we perform analysis and empirical studies on noise properties of images reconstructed from parallel-beam, fan-beam, and cone-beam data. Finally, a discussion is given in Section IV.

II. CHORD-BASED RECONSTRUCTION ALGORITHMS

We consider a continuous source trajectory specified by $\vec{r}_0(s) = (x(s), y(s), z(s))$, where $x(s)$, $y(s)$, and $z(s)$ denote the x -, y -, and z - components of $\vec{r}_0(s)$ in the fixed-coordinate system, and s is a curve parameter indicating the position of the X-ray source on the trajectory. The projection data of the object function $f(\vec{r})$ can be mathematically expressed as

$$D(\vec{r}_0(s), \hat{\beta}) = \int_0^\infty dt f(\vec{r}_0(s) + t\hat{\beta}) \quad (1)$$

where the unit vector $\hat{\beta}$ denotes the direction of a specific X-ray passing through the point $\vec{r}$. We also introduce two additional coordinate systems $\{u, v, w\}$ and $\{u_d, v_d\}$ to describe the geometry in a general scan. They are fixed on the rotating source point and the cone-beam projection of the source point, respectively, which are referred to as the rotation-coordinate and detector-coordinate systems. Let $\hat{e}_u(s)$, $\hat{e}_v(s)$, and $\hat{e}_w(s)$ denote the orthogonal unit vectors of the rotation-coordinate system. The rotation-coordinate system can be chosen such that $\hat{e}_u(s)$ and $\hat{e}_w(s)$ are within the $x-y$ plane and $\hat{e}_v(s)$ is parallel to the z axis. One can also choose the “well oriented” coordinate system as the rotation-coordinate system [5], in which unit vector $\hat{e}_u(s)$ is parallel to and unit vectors $\hat{e}_v(s)$ and $\hat{e}_w(s)$

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are perpendicular to the direction of $(d\vec{r}_0(s)/ds)$. We assume that a detector plane is placed at a distance S from the source point and orients along $\hat{e}_w(s)$. The detector-coordinate system $\{u_d, v_d\}$ is the cone-beam projection of the 2-D coordinate system $\{u, v\}$ onto the detector plane, and the u_d and v_d axis are along $\hat{e}_u(s)$ and $\hat{e}_v(s)$, respectively. In this situation, we also use $P(u_d, v_d, s)$ to denote the cone-beam projection, thus $D(\vec{r}_0(s), \hat{\beta}) = P(u_d, v_d, s)$, when

$$\hat{\beta} = \frac{1}{A(u_d, v_d)} [u_d \hat{e}_u(s) + v_d \hat{e}_v(s) - S \hat{e}_w(s)]$$

and $A(u_d, v_d) = \sqrt{u_d^2 + v_d^2 + S^2}$. (2)

In a 2-D case, it can be observed that $v_d = 0$. For notational convenience, we use $A(u_d)$ and $P(u_d, s)$ to denote $A(u_d, 0)$ and $P(u_d, 0, s)$, respectively.

A *chord* is a line segment connecting two points $\vec{r}_0(s_a)$ and $\vec{r}_0(s_b)$ on the trajectory. Any point $\vec{r}$ on the chord can be expressed as

$$\vec{r} = \frac{\vec{r}_0(s_a) + \vec{r}_0(s_b)}{2} + x_c \hat{e}_c, \quad x_c \in [-l, l] \quad (3)$$

where $\hat{e}_c = (\vec{r}_0(s_b) - \vec{r}_0(s_a)) / |\vec{r}_0(s_b) - \vec{r}_0(s_a)|$ denotes the direction of the chord, and $l = (1/2)|\vec{r}_0(s_b) - \vec{r}_0(s_a)|$ is one half of the chord length. For a helical trajectory, the curve parameter s is linearly related to the rotation angle λ , and in the current work, we select $s = \lambda$. When s_a and s_b are within one turn, the chord becomes the conventional π -line segment [2], [11], [12]. The intersection between a chord and the object is referred to as a *support segment*. Let x_{c1} and x_{c2} represent the end points of a support segment. Because the trajectory under consideration never intersects the object, we have $[x_{c1}, x_{c2}] \subset [-l, l]$. Therefore, one can use (x_c, s_a, s_b) and $f_c(x_c, s_a, s_b)$ to denote a point and the corresponding image on the chord. We have previously developed three algorithms, which are referred to as the BPF [2], [6], [10], MDFBP [4], [6], and FBP [6], [10] algorithms, respectively, for exact image reconstruction on a chord of a general trajectory.

A. BPF Algorithm

The BPF algorithm [2], [6] reconstructs the image on a chord specified by s_a and s_b as

$$f_c(x_c, s_a, s_b) = \hat{f}(x_c, s_a, s_b) + \frac{1}{2\pi} \frac{P(u_{d0}, v_{d0}, s_a)}{b(x_c)} \times \left[\frac{\sqrt{(l-x_B)(l-x_A)}}{l-x_c} + \frac{\sqrt{(l+x_A)(l+x_B)}}{l+x_c} \right] \quad (4)$$

where $x_c \in [x_A, x_B]$, and parameters x_A and x_B are two points on the chord satisfying $[x_{c1}, x_{c2}] \subseteq [x_A, x_B] \subset [-l, l]$. The function $b(x_c)$ is defined as $b(x_c) = \sqrt{(x_B - x_c)(x_c - x_A)}$, and $P(u_{d0}, v_{d0}, s_a)$ denotes the projection along the chord

specified by s_a and s_b . The filtered image $\hat{f}(x_c, s_a, s_b)$ is given by

$$\hat{f}(x_c, s_a, s_b) = \frac{1}{2\pi^2} \frac{1}{b(x_c)} \int_{\mathbb{R}} \frac{dx'_c}{x_c - x'_c} b(x'_c) g(x'_c, s_a, s_b) \quad (5)$$

where the backprojection image on the chord can be written as

$$g(x'_c, s_a, s_b) = \Pi_c(x'_c) \int_{s_a}^{s_b} ds \frac{\text{sgn}(-\hat{\beta} \cdot \hat{e}_w)}{|\vec{r}' - \vec{r}_0(s)|^2} \times \left\{ -\frac{d\vec{r}_0(s)}{ds} \cdot \hat{\beta} P(u_d, v_d, s) + \left[\frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_u(s) + \frac{u_d}{S(s)} \frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_w(s) \right] \times A(u_d, v_d) \frac{\partial P(u_d, v_d, s)}{\partial u_d} + \left[\frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_v(s) + \frac{v_d}{S(s)} \frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_w(s) \right] \times A(u_d, v_d) \frac{\partial P(u_d, v_d, s)}{\partial v_d} \right\} \quad (6)$$

and the rect function $\Pi_c(x'_c) = 1$ if $x'_c \in [x_A, x_B]$ and zero otherwise. It can be observed in (4) that the chord image can be obtained exactly from knowledge of the backprojection image $g(x'_c, s_a, s_b)$ for $x'_c \in [x_A, x_B]$, which we refer to as the *reconstruction segment* because it determines the actual reconstruction interval on the chord. In particular, because the reconstruction segment $[x_A, x_B]$ can be chosen as small as the support segment $[x_{c1}, x_{c2}]$, the chord image can be reconstructed from knowledge of $g(x'_c, s_a, s_b)$ only on the support segment. This interesting property of the Hilbert transform forms the basis for exact image reconstruction on a chord from projections containing longitudinal or transverse truncations [13].

B. MDFBP Algorithm

The BPF algorithm reconstructs the chord image by performing a 1-D filtration [i.e., the integration over x'_c in (4)] of the backprojection image [i.e., the integration over s in (6)]. On the other hand, the MDFBP algorithm [4], [6] reconstructs the chord image by performing a 1-D data filtration (i.e., the integration over u'_c) prior to their backprojection (i.e., the integration over s) onto the chord

$$f_c(x_c, s_a, s_b) = \frac{1}{2\pi^2} \frac{1}{b(x_c)} \int_{s_a}^{s_b} ds [w_2(1 - u_c) + w_1 u_c] \int_{\mathbb{R}} \frac{du'_c}{u_c - u'_c} P_{\Pi} + \frac{1}{2\pi} \frac{P(u_{d0}, v_{d0}, s_a)}{b(x_c)} \times \left[\frac{\sqrt{(l-x_B)(l-x_A)}}{l-x_c} + \frac{\sqrt{(l+x_A)(l+x_B)}}{l+x_c} \right] \quad (7)$$

where the modified data function is given by

$$P_{\Pi} = \Pi_c(x'_c) \frac{b(x'_c)}{w_2(1 - u'_c) + w_1 u'_c} \frac{\text{sgn}(-\hat{\beta} \cdot \hat{e}_w)}{|\vec{r} - \vec{r}_0(s)|^2} \\ \times \left\{ -\frac{d\vec{r}_0(s)}{ds} \cdot \hat{\beta} P(u_d, v_d, s) \right. \\ \left. + \left[\frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_u(s) + \frac{u_d}{S(s)} \frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_w(s) \right] \right. \\ \times A(u_d, v_d) \frac{\partial P(u_d, v_d, s)}{\partial u_d} \\ \left. + \left[\frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_v(s) + \frac{v_d}{S(s)} \frac{d\vec{r}_0(s)}{ds} \cdot \hat{e}_w(s) \right] \right. \\ \left. \times A(u_d, v_d) \frac{\partial P(u_d, v_d, s)}{\partial v_d} \right\}$$

$w_1 = -[\vec{r}_0(s_a) - \vec{r}_0(s)] \cdot \hat{e}_w$, and $w_2 = -[\vec{r}_0(s_b) - \vec{r}_0(s)] \cdot \hat{e}_w$. For a source position s , the variables u_c and u'_c denote the cone-beam projections of x_c and x'_c onto the detector and can be obtained, respectively, by replacing x with x_c and x'_c in

$$u = \frac{w_2(x + l)}{w_1(l - x) + w_2(x + l)}. \quad (8)$$

The rect function $\Pi_c(x'_c)$ in (8) indicates that the MDFBP algorithm can reconstruct a chord image from knowledge of data only on the cone-beam projection of the reconstruction segment $[x_A, x_B]$, which can be as small as the support segment. Therefore, similar to the BPF algorithm, the MDFBP algorithm can also reconstruct a chord image from data containing truncations [4], [6].

C. FBP Algorithm

The chord-based FBP algorithm [6], [10] can be expressed as

$$f_c(x_c, s_a, s_b) = \frac{1}{2\pi^2} \int_{s_a}^{s_b} ds \frac{A}{|\vec{r} - \vec{r}_0(s)|} \\ \times \int_{-\infty}^{\infty} \frac{du'_c}{u_c - u'_c} \frac{1}{|\vec{r} - \vec{r}_0(s)|} \frac{\partial}{\partial q} D(\vec{r}_0(q), \hat{\beta}) \Big|_{q=s} \quad (9)$$

where u_c indicates the cone-beam projection of x_c onto the detector and is determined by using x_c to replace x in (8), and A denotes the distance from the source point $\vec{r}_0(s)$ to a point on the detector at which the ray connecting $\vec{r}$ and $\vec{r}_0(s)$ intersects the detector. As the filtering (i.e., the integration over u'_c) is carried out over the projection of the straight line containing the chord, similar to other existing FBP-based algorithms, the chord-based FBP algorithm cannot exactly reconstruct ROI images from data containing transverse truncations.

D. Data-Sufficiency Conditions

As shown in (9), a data-sufficiency condition for the FBP algorithm is: 1) data are available over the trajectory segment $s \in [s_a, s_b]$ and 2) for each s , data on the cone-beam projection of the chord are nontruncated. This condition is similar to that for other FBP-based algorithms [1], [9], [14], [15], [16]. From (4) and (7), a data-sufficiency condition for the chord-based

BPF and MDFBP algorithms is: 1) data are collected over the trajectory segment $[s_a, s_b]$ and 2) at each s , data only on the cone-beam projection of the reconstruction segment $[x_A, x_B]$ on the chord are available. It follows that, because the reconstruction segment $[x_A, x_B]$ can be chosen as small as the support segment $[x_{c1}, x_{c2}]$, the BPF and MDFBP algorithms require, at each s , data only on the cone-beam projection of the support segment $[x_{c1}, x_{c2}]$ (instead of the entire chord-line as the chord-based FBP algorithm requires). Different selections of the reconstruction segment $[x_A, x_B]$ imply that different amounts of data at each s can be used for reconstructing the chord image. Under the ideal continuous conditions, different selections of $[x_A, x_B]$ yield identical chord images. However, when data contain noise and other inconsistencies, and when different selections of $[x_A, x_B]$ are used, the BPF and MDFBP algorithms in their discrete forms may yield different chord images. This is an issue that will be investigated in the following.

III. NOISE PROPERTIES OF CHORD-BASED IMAGE RECONSTRUCTION

The BPF, MDFBP, and FBP algorithms described above can be applied to reconstructing chord images from parallel-, fan-, and cone-beam data [17]. Algorithms analogous to the BPF algorithm that are capable of reconstructing 2-D ROI images from truncation data have also previously been proposed [8], [13], [18]. We study the noise properties of chord-based reconstruction by use of these algorithms in their discrete forms. As mentioned above, the BPF and MDFBP algorithms can reconstruct the image on the reconstruction segment $[x_A, x_B]$ as long as it covers the support segment $[x_{c1}, x_{c2}]$. We analyze image-noise properties on reconstruction segments of different lengths.

A. Analysis of Image-Noise Properties

The chord-based algorithms invoke three major mathematical operations: differentiation, backprojection, and filtration. To a large extent, the BPF, MDFBP, and FBP algorithms differ in the orders of invoking these operations. Below, we focus on investigating the noise properties of differentiation, backprojection, and filtration in the BPF algorithm. The approach taken in the investigation is readily applicable to analyzing the noise properties of the MDFBP and FBP algorithms. In the presence of data noise, the measured projection $\mathbf{D}(\vec{r}_0(s), \hat{\beta})$ should be interpreted as a stochastic process. (Throughout the paper, we use boldface and normal letters to denote a stochastic process and its mean, respectively.) Because the backprojection $\mathbf{g}(x_c, s_a, s_b)$ and the final image $\mathbf{f}_c(x_c, s_a, s_b)$ on a chord are computed from $\mathbf{D}(\vec{r}_0(s), \hat{\beta})$, they should also be considered as stochastic processes. We focus in this section on investigating the chord-image variance $\mathbf{f}_c(x_c, s_a, s_b)$ by the investigation of noise propagation through each step involved in the BPF reconstruction algorithm.

1) *Noise Properties of Differentiation/Backprojection for Parallel-Beam Data:* Let $\mathbf{P}(u_d, s)$ denote the parallel-beam projection on detector bin u_d acquired at view s . We assume the covariance of the projection data $\mathbf{P}(u_d, s)$ to be uncorrelated, i.e.,

$$\text{Cov}\{\mathbf{P}(u_d, s), \mathbf{P}(u'_d, s')\} = \sigma^2(u_d, s) \delta(u_d - u'_d) \delta(s - s') \quad (10)$$

where $\sigma^2(u_d, s)$ denotes the variance of the projection data. The backprojection image on the chord is given by [13]

$$\mathbf{g}(x_c, s_a, s_b) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \frac{\partial}{\partial u_d} \mathbf{P}(u_d, s) \quad (11)$$

where $u_d = \vec{r} \cdot \hat{e}_u(s)$. The final image variances on a chord depend upon the covariance of the backprojection image, which, using (11), can be written as

$$\begin{aligned} & \text{Cov} \{ \mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b) \} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds' \text{Cov} \left\{ \frac{\partial}{\partial u_d} \mathbf{P}(u_d, s), \frac{\partial}{\partial u'_d} \mathbf{P}(u'_d, s') \right\}. \quad (12) \end{aligned}$$

The evaluation of the backprojection-image covariance involves the data-derivative covariance, which can be conveniently written as

$$\begin{aligned} & \text{Cov} \left\{ \frac{\partial}{\partial u_d} \mathbf{P}(u_d, s), \frac{\partial}{\partial u'_d} \mathbf{P}(u'_d, s') \right\} = a\omega\sigma^2(u_d, s) \\ & \quad \times \delta(u_d - u'_d) \delta(s - s') + T_{\text{para}}(u_d, u'_d, s, s') \quad (13) \end{aligned}$$

where $\text{Var}\{\mathbf{P}(u_d, s)\} = \sigma^2(u_d, s)$ denotes the known data variance, which can be a function of u_d and s . The second term $T_{\text{para}}(u_d, u'_d, s, s')$ represents the difference between the term on the left-hand side and the first term on the right-hand side of (13). Although the magnitude of $T_{\text{para}}(u_d, u'_d, s, s')$ can be larger than or comparable to that of the first term in the right-hand side of (13), numerical results that follow show that its contribution to the final image variance on a chord is negligibly small. Therefore, we consider only the first term in the derivation of the chord-image variance below. The parameters a and ω are introduced to account for the interpolation effect of the discrete data derivative and discrete backprojection on the chord-image variance. Using the first term in (13), we can rewrite (12) as

$$\begin{aligned} & \text{Cov} \{ \mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b) \} \\ & \approx a\omega \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \sigma^2(u_d, s) \delta(u_d - u'_d). \quad (14) \end{aligned}$$

We now consider two points x_c and x'_c on the chord and let u_d and u'_d denote their parallel-beam or fan-beam projections onto the detector. Clearly, for a source position s that satisfies $s \neq s_a$ or s_b , one can conclude that

$$\begin{aligned} u_d - u'_d &= 0 & \text{if } x_c - x'_c &= 0 \\ u_d - u'_d &\neq 0 & \text{if } x_c - x'_c &\neq 0. \end{aligned} \quad (15)$$

Thus, if $x_c = x'_c$, $\text{Cov}\{\mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x_c, s_a, s_b)\} = \delta(0)a\omega \int_{-\pi/2}^{\pi/2} ds \sigma^2(u_d, s)$ and, if $x_c \neq x'_c$,

$\text{Cov}\{\mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b)\} = 0$. Therefore, the covariance of the backprojection image for parallel-beam projections can be re-expressed as

$$\text{Cov} \{ \mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b) \} \approx c(x_c) \delta(x_c - x'_c) \quad (16)$$

where

$$c(x_c) = a\omega \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \sigma^2(u_d, s). \quad (17)$$

2) *Noise Properties of Differentiation/Backprojection for Fan-Beam Data:* In the fan-beam case, we use $\mathbf{P}(u_d, s)$ to denote the projection on detector bin u_d acquired at view s . Again, we assume $\mathbf{P}(u_d, s)$ to be uncorrelated and to satisfy (10). The backprojection image in (6) can be re-expressed as

$$\mathbf{g}(x_c, s_a, s_b) = \int_{s_a}^{s_b} ds \frac{1}{|\vec{r} - \vec{r}_0(s)|^2} \mathbf{P}'(u_d, s) \quad (18)$$

where $u_d = (S\vec{r} \cdot \hat{e}_u(s)) / (\vec{r}_0(s) - \vec{r}) \cdot \hat{e}_w(s)$ is the fan-beam projection of x_c onto the detector. The weighted-data derivative $\mathbf{P}'(u_d, s)$ is given by

$$\mathbf{P}'(u_d, s) = A^2(u_d) \left| \frac{d\vec{r}_0(s)}{ds} \right| \frac{\partial}{\partial u_d} \left[\frac{\mathbf{P}(u_d, s)}{A(u_d)} \right]. \quad (19)$$

Using (18), one can write the covariance of the backprojection image as

$$\begin{aligned} & \text{Cov} \{ \mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b) \} \\ &= \int_{s_a}^{s_b} ds \int_{s_a}^{s_b} ds' \frac{1}{|\vec{r} - \vec{r}_0(s)|^2} \frac{1}{|\vec{r} - \vec{r}_0(s')|^2} \\ & \quad \times \text{Cov} \{ \mathbf{P}'(u_d, s), \mathbf{P}'(u'_d, s') \} \quad (20) \end{aligned}$$

which depends upon the covariance of $\mathbf{P}'(u_d, s)$. Again, we can conveniently write the covariance of $\mathbf{P}'(u_d, s)$ as

$$\begin{aligned} & \text{Cov} \{ \mathbf{P}'(u_d, s), \mathbf{P}'(u'_d, s') \} \\ &= a\omega\sigma^2(u_d, s) A^2(u_d) \left| \frac{d\vec{r}_0(s)}{ds} \right|^2 \delta(u_d - u'_d) \delta(s - s') \\ & \quad + T_{\text{fan}}(u_d, u'_d, s, s') \quad (21) \end{aligned}$$

where $\text{Var}\{\mathbf{P}(u_d, s)\} = \sigma^2(u_d, s)$ denotes the known data variance, which can be a function of u_d and s . The second term $T_{\text{fan}}(u_d, u'_d, s, s')$ represents the difference between the term on the left-hand side and the first term in the right-hand side of (21). As numerical results indicate, it turns out that $T_{\text{fan}}(u_d, u'_d, s, s')$ will also have a negligible contribution to the chord-image variance. Therefore, we consider only the first term in the derivation of the chord-image variance below. Again, the parameters a and ω are introduced to account for the interpolation effect of the discrete data derivative and discrete

backprojection on the chord-image variance. Using the first term in (21), we can rewrite (20) as

$$\text{Cov}\{\mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b)\} \approx a\omega \int_{s_a}^{s_b} ds \frac{\sigma^2(u_d, s) A^2(u_d)}{|\vec{r} - \vec{r}_0(s)|^4} \left| \frac{d\vec{r}_0(s)}{ds} \right|^2 \delta(u_d - u'_d). \quad (22)$$

Similar to the parallel-beam case described previously, using (15), one can conclude that

$$\text{Cov}\{\mathbf{g}(x_c, s_a, s_b), \mathbf{g}(x'_c, s_a, s_b)\} \approx c(x_c) \delta(x_c - x'_c) \quad (23)$$

where

$$c(x_c) = a\omega \int_{s_a}^{s_b} ds \frac{A^2(u_d)}{|\vec{r} - \vec{r}_0(s)|^4} \left| \frac{d\vec{r}_0(s)}{ds} \right|^2 \sigma^2(u_d, s). \quad (24)$$

3) Noise Properties of Differentiation/Backprojection for Cone-Beam Data: In the cone-beam case, let $\mathbf{P}(u_d, v_d, s)$ denote the projection at view s on a detector bin specified by (u_d, v_d) . In the so-called “well oriented” rotation-coordinate system [5], unit vector $\hat{e}_u(s)$ is parallel to and unit vectors $\hat{e}_v(s)$ and $\hat{e}_w(s)$ are orthogonal to the tangential direction $(d\vec{r}_0(s)/ds)$ of the source trajectory. Let u_d and v_d denote the coordinates along $\hat{e}_u(s)$ and $\hat{e}_v(s)$. It can be shown [5] that the backprojection image depends only upon the data derivative along u_d . Therefore, the reconstruction formula for the cone-beam backprojection image can be obtained from that for the fan-beam backprojection image in (18) by simply replacing $P(u_d, s)$ and $A(u_d)$ with $P(u_d, v_d, s)$ and $A(u_d, v_d)$, respectively. Subsequently, one can show that the covariance of the cone-beam backprojection image $\mathbf{g}(x_c, s_a, s_b)$ also satisfies (23) and (24).

4) Estimation of Parameters a and ω in Discrete Form: The parameter a is introduced to account for the interpolation effect of the discrete data-derivative on the chord-image variance. We consider a two-point derivative, which was used in our numerical studies. Let $\mathbf{P}_i$ denote the discrete data, where $i = 1, 2, \dots, I$, and I indicates the total number of data points. We assume that data $\mathbf{P}_i$ are uncorrelated and use $\text{Var}\{\mathbf{P}_i\}$ to denote their variances. The two-point data derivative is defined as

$$\bar{\mathbf{P}}_i = \frac{1}{2}[\mathbf{P}_{i+1} - \mathbf{P}_{i-1}]. \quad (25)$$

Therefore, the variance of the discrete data derivative $\bar{\mathbf{P}}_i$ can be written as

$$\text{Var}\{\bar{\mathbf{P}}_i\} = \frac{1}{2} \frac{\text{Var}\{\mathbf{P}_{i+1}\} + \text{Var}\{\mathbf{P}_{i-1}\}}{2}. \quad (26)$$

When data variances are identical, (26) becomes

$$\text{Var}\{\bar{\mathbf{P}}_i\} = \frac{1}{2} \text{Var}\{\mathbf{P}_i\}. \quad (27)$$

Therefore, in our studies, we select $a = 1/2$, which is the coefficient in (27).

The parameter ω was introduced to account for the interpolation effect of discrete backprojection on the chord-image variances. The estimated value of ω depends obviously upon the

specific interpolation scheme used in the discrete backprojection. We illustrate our estimation of ω when a two-point interpolation is used for the discrete backprojection in the parallel- and fan-beam cases. One can readily obtain estimates of ω when other interpolation schemes are used. In our studies, u_d is measured in the unit of detector-bin size. Let $i = \text{floor}(u_d) + I_0$, where I_0 denotes the index corresponding to $u_d = 0$. At a backprojection view s , we use $\mathbf{P}'_i$ to denote the discrete weighed-data derivatives. For a given u_d , we express the interpolated weighed-data derivative as

$$\mathbf{P}'_{u_d} = (1 - \gamma)\mathbf{P}'_i + \gamma\mathbf{P}'_{i+1} \quad (28)$$

where $\gamma = u_d - i$. Furthermore, we can write the variance of $\mathbf{P}'_{u_d}$ as

$$\text{Var}\{\mathbf{P}'_{u_d}\} \approx [(1 - \gamma)^2 + \gamma^2] \text{Var}\{\mathbf{P}'_i\}. \quad (29)$$

For the sake of simplifying the estimation of ω , we have ignored the correlation between $\mathbf{P}'_i$ and $\mathbf{P}'_{i+1}$ and assumed that $\text{Var}\{\mathbf{P}'_i\} \approx \text{Var}\{\mathbf{P}'_{i+1}\}$. We select ω as the average over all of the possible γ s, which can be computed as

$$\omega = \int_0^1 [(1 - \gamma)^2 + \gamma^2] d\gamma = \frac{2}{3}. \quad (30)$$

Finally, with a substitution of $a = 1/2$ and $\omega = 2/3$ into (17) and (24), we obtain the variances $c(x_c)$ of the backprojection images on the chords for the parallel-beam and fan-beam projections, respectively, as

$$c(x_c) = \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \sigma^2(u_d, s) \quad (31)$$

$$c(x_c) = \frac{1}{3} \int_{s_a}^{s_b} ds \frac{A^2(u_d)}{|\vec{r} - \vec{r}_0(s)|^4} \left| \frac{d\vec{r}_0(s)}{ds} \right|^2 \sigma^2(u_d, s). \quad (32)$$

5) Noise Property of Weighted Hilbert Transform Over a Finite Interval: The weighted Hilbert transform constitutes an important step in the chord-based BPF, MDFBP, and FBP algorithms. Consequently, the noise properties of these algorithms depend upon that of the weighted Hilbert transform, which we study in the following. Let $\hat{f}(x_c, s_a, s_b)$ denote the weighted Hilbert transform of the backprojection image $g(x_c, s_a, s_b)$

$$\hat{f}(x_c, s_a, s_b) = \frac{1}{b(x_c)} \int_{x_A}^{x_B} \frac{dx'_c}{x_c - x'_c} b(x'_c) g(x'_c, s_a, s_b). \quad (33)$$

We assume that $g(x'_c, s_a, s_b)$ is band-limited to ν_m . Therefore, the Hilbert transform kernel $(1/x_c)$ can be replaced by

$$h(x_c) = -\pi j \int_{-\nu_m}^{\nu_m} d\nu \text{sgn}[\nu] e^{2\pi j \nu x_c} = \frac{2\sin^2(\pi \nu_m x_c)}{x_c}. \quad (34)$$

In the presence of noise, the weighted Hilbert transform $\hat{f}(x_c, s_a, s_b)$ should be interpreted as a stochastic process,

which is denoted in boldface. The variance of $\hat{\mathbf{f}}(x_c, s_a, s_b)$ can be written as

$$\begin{aligned} \text{Var} \left\{ \hat{\mathbf{f}}(x_c, s_a, s_b) \right\} &= \frac{1}{b^2(x_c)} \int_{x_A}^{x_B} dx'_c \int_{x_A}^{x_B} dx''_c b(x'_c) b(x''_c) \\ &\quad \times h(x_c - x'_c) h(x_c - x''_c) \\ &\quad \times \text{Cov} \left\{ \mathbf{g}(x'_c, s_a, s_b), \mathbf{g}(x''_c, s_a, s_b) \right\}. \end{aligned} \quad (35)$$

As (16) and (23) show, the backprojection image $\mathbf{g}(x_c, s_a, s_b)$ can be treated as an approximated uncorrelated stochastic process. Using the result of (16) or (23), we can write (35) as

$$\begin{aligned} \text{Var} \left\{ \hat{\mathbf{f}}(x_c, s_a, s_b) \right\} &= \frac{1}{b^2(x_c)} \int_{x_A}^{x_B} dx'_c c(x'_c) h^2(x_c - x'_c) b^2(x'_c). \end{aligned} \quad (36)$$

In our numerical studies in Sections III-B1 and III-C1, we have used $\nu_m = (1/2\Delta_c)$, where Δ_c denote the sample interval of $\mathbf{g}(x_c, s_a, s_b)$.

6) *Noise Properties of Chord Images:* Using (4), one can write the variance of the reconstructed chord image as

$$\begin{aligned} \text{Var} \{ \mathbf{f}_c(x_c, s_a, s_b) \} &\approx \text{Var} \left\{ \hat{\mathbf{f}}(x_c, s_a, s_b) \right\} + \frac{\text{Var} \{ \mathbf{P}(u_{d0}, v_{d0}, s_a) \}}{4\pi^2 b^2(x_c)} \\ &\quad \times \left[\frac{\sqrt{(l-x_B)(x_A+l)}}{l-x_c} + \frac{\sqrt{(l+x_A)(l+x_B)}}{l+x_c} \right]^2. \end{aligned} \quad (37)$$

Substituting (36) into (37) yields

$$\begin{aligned} \text{Var} \{ \mathbf{f}_c(x_c, s_a, s_b) \} &\approx \frac{1}{b^2(x_c)} \int_{x_A}^{x_B} dx'_c c(x'_c) h^2(x_c - x'_c) b^2(x'_c) \\ &\quad + \frac{\text{Var} \{ \mathbf{P}(u_{d0}, v_{d0}, s_a) \}}{4\pi^2 b^2(x_c)} \\ &\quad \times \left[\frac{\sqrt{(l-x_B)(x_A+l)}}{l-x_c} + \frac{\sqrt{(l+x_A)(l+x_B)}}{l+x_c} \right]^2 \end{aligned} \quad (38)$$

which provides a formula for computing the chord-image variance.

B. Numerical Studies of Noise Properties in Parallel-Beam Reconstruction

Using the parallel-beam configuration in Fig. 1(b) and the parameters given in Table I, we calculated noiseless projections for the numerical phantom in Fig. 1(a). We have used an object-independent Gaussian noise model and an object-dependent Poisson-noise model in the numerical studies. For each noise model, we generated 10 000 sets of noisy data by using noiseless data as the means. The standard deviation σ_0 of Gaussian noise used is 1.6% of the maximum value in the noiseless data,

Fig. 1. (a) Phantom in numerical studies. (b) Parallel-beam configuration. Solid line segment with endpoints x_A and x_B represents reconstruction segment. Thick segment with endpoints x_{c1} and x_{c2} indicates support segment. It should be noted that $[x_A, x_B] \supseteq [x_{c1}, x_{c2}]$. Rectangular ROI is decomposed into a set of (dashed) line segments.

whereas the standard deviation for the Poisson noise is the noiseless data scaled to yield a total photon count of 5×10^5 for each view. We investigated four reconstruction segments with different lengths $L_{AB} = |x_B - x_A|$: 7.8, 10.0, 14.1, and 20.0 cm, all of which are located at $x = 4.06$ cm. It can be observed in Fig. 1(b) that the length of the support segment, 5.5 cm in length, is shorter than the four reconstruction segments considered. Therefore, the image on this chord can be reconstructed exactly by use of data determined by these reconstruction segments. One can also conclude from Fig. 1(b) that the minimum data required by the first three reconstruction segments, which are shorter than the maximum dimension (about 15.6 cm) of the object support, are truncated.

1) *Noise Properties in Reconstruction from Truncated Parallel-Beam Data:* From the 10 000 sets of data containing Gaussian noise, we used (4)–(6) to reconstruct 10 000 noisy $\mathbf{f}_c(x_c, s_a, s_b)$, $\hat{\mathbf{f}}(x_c, s_a, s_b)$, and $\mathbf{g}(x_c, s_a, s_b)$, respectively, on the four reconstruction segments described previously. Based upon these noisy reconstructions, we subsequently computed their corresponding empirical variances, which are shown in the upper row of Fig. 2. We compare the empirical results to the analytical results obtained by use of (16), (36), and (38). The function $c(x_c)$ is determined by using $\sigma(u_d, s) = \sigma_0$ in (31), where σ_0 is 1.6% of the maximum value in noiseless data. The analytical results are displayed in the middle row of Fig. 2. Similarly, using (4)–(6), we reconstructed 10 000 sets of noisy $\mathbf{g}(x_c, s_a, s_b)$, $\hat{\mathbf{f}}(x_c, s_a, s_b)$, and $\mathbf{f}_c(x_c, s_a, s_b)$ on the four segments from 10 000 sets of data containing Poisson noise. The computed empirical variances from these noisy images are displayed in the upper row of Fig. 3. Using the noiseless data as the Poisson-noise variance $\sigma^2(u_d, s)$ in (17), one can readily determine $c(x_c)$; and using the determined $c(x_c)$ in (16), (36), and (38), one can compute analytical image variances, which are displayed in the middle row of Fig. 3. The results show that the analytical and empirical results agree well with each other, suggesting that (38) provides an adequate estimation of the chord-image variance.

It can also be observed in Fig. 2(c) and Fig. 3(c) that the shorter the reconstruction segment, the higher the chord-image variances. This is only because the second term in (38) increases as L_{AB} (i.e., $(x_A - x_c)(x_c - x_B)$) decreases. However, the difference of the chord-image variances in the central part of the support segment is quite small among these reconstruction segments. The implication of this result is that there may be a significant gain in terms of dose reduction by using a short

TABLE I
PARAMETERS FOR CIRCULAR PARALLEL-BEAM, CIRCULAR FAN-BEAM, AND HELICAL CONE-BEAM SCANNING CONFIGURATIONS WHICH ARE USED IN OUR SIMULATION STUDIES. SDD IS SOURCE-TO-DETECTOR DISTANCE AND SID DENOTES SOURCE-TO-ISOCENTER DISTANCE

	Circular parallel-beam	Circular fan-beam	Helical cone-beam
	scan	scan	scan
Number of views per turn	1024	512	300
Angular range	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-\pi, \pi]$	$[-\pi, \pi]$
Detector bins	256	256	128×128
Detector bin size (mm)	1.1	1.6	3.1×3.1
SDD (mm)	-	200	200
SID (mm)	-	200	200
Helical pitch (mm)	-	-	90

Fig. 2. Empirical (top row) and analytical (middle row) variances of (a) $\mathbf{g}(x_c, s_a, s_b)$, (b) $\hat{\mathbf{f}}(x_c, s_a, s_b)$, and (c) $\mathbf{f}_c(x_c, s_a, s_b)$ obtained on four reconstruction segments from parallel-beam data containing Gaussian noise. Difference between empirical variances and analytical variances is also shown in bottom row, which demonstrates analytical variances agree well with the empirical variances. Lengths of these segments are indicated in the box in upper-right corners of plots.

reconstruction segment, because data required to reconstruct an image on this reconstruction segment is less than that required by using a longer reconstruction segment, thus resulting in a reduced illumination coverage to the object. For similar X-ray

intensities, which are directly related to the data-noise level, the reconstruction using a short reconstruction segment appears to yield image variance within the support segment that is comparable to that obtained with a longer reconstruction segment.

Fig. 3. Empirical (top row) and analytical (middle row) variances of (a) $\mathbf{g}(x_c, s_a, s_b)$, (b) $\hat{\mathbf{f}}(x_c, s_a, s_b)$, and (c) $\mathbf{f}_c(x_c, s_a, s_b)$ obtained on four reconstruction segments from parallel-beam data containing Poisson noise. Difference between empirical variances and analytical variances is also shown in bottom row, which demonstrates analytical variances agree well with empirical variances. Lengths of these segments are indicated in the box in upper-right corners of plots.

We have also performed numerical studies of the noise properties of the reconstructed ROI images by use of the BPF and MDFBP algorithms from truncated data. Using the numerical phantom in Fig. 1 and each of Gaussian- and Poisson-noise models described above, we generated 500 noisy, truncated data sets for image reconstruction on reconstruction segments of a length $L_{AB} = 10.0$ cm, as shown in Fig. 1(a), which completely cover the ROI. We subsequently reconstructed 500 noisy images by using the BPF and MDFBP algorithms. We display in Fig. 4 noisy ROI images reconstructed using the BPF and MDFBP algorithms from data containing Gaussian noise (upper row) and Poisson noise (lower row).

Using the reconstructed 500 sets of Gaussian-noise images and 500 sets of Poisson-noise images, we computed empirical variances of the ROI images, which are shown in the upper row and lower row of Fig. 5, respectively. We display in the third column of Fig. 5 the variance profiles on the dashed lines indicated in the variance images. Results in Fig. 5 support the conclusion that both BPF and MDFBP algorithms yield images with comparable variance levels.

Fig. 4. ROI-images obtained by use of (a) BPF algorithm and (b) MDFBP algorithm from truncated parallel-beam data containing Gaussian noise (upper row) and Poisson noise (lower row).

Fig. 5. Empirical ROI-image variances obtained by use of (a) BPF and (b) MDFBP algorithms from truncated parallel-beam data containing Gaussian noise (upper row) and Poisson noise (lower row), respectively. (c) Variance profiles on the dashed lines indicated in columns (a) and (b) obtained with the BPF (solid) and MDFBP (dashed) algorithms, respectively. For purpose of displaying details in central (i.e., low variance) regions, we have applied a logarithmic scale to the variance images. Display windows are $[-0.86, 0.50]$ and $[-1.28, 0.20]$ for Gaussian noise and Poisson noise, respectively.

2) *Noise Properties in Reconstruction from Nontruncated Parallel-Beam Data:* As discussed above, the FBP algorithm cannot reconstruct exactly images from truncated data. Therefore, we study the noise properties of the FBP algorithm from parallel-beam data without truncations. For the purpose of comparison, we have also included reconstruction results of the BPF and MDFBP algorithms from the same nontruncated data. Using the numerical phantom in Fig. 1 and each of the Gaussian- and Poisson-noise models, we generated 500 noisy data sets from which 500 noisy images were obtained by use of each of the BPF, MDFBP, and FBP algorithms. Using these noisy images, we computed empirical variance images, which are shown in the upper and lower rows of Fig. 6, respectively, for the Gaussian- and Poisson-noise models. We also display in Fig. 7 the variance profiles on the dashed lines (i.e., on a chord) indicated on the variance images in Fig. 6. The profile results were obtained by use of the BPF (solid), MDFBP (dashed), and FBP (dotted) algorithms for the (a) Gaussian-noise model and (b) Poisson-noise model. It can be observed that image variances obtained with the three algorithms are similar and that the only difference comes at the extreme ends of the shown reconstruction segments. The BPF and MDFBP algorithms show a significant increase in the image variance at both ends of the profile. The reason for this is that the reconstruction segment was taken to be the width of the image array, and the prefactor for the finite Hilbert transform in (4) and (7) has a singularity at the ends of the reconstruction segment. In practical situations this prefactor is of little consequence because the reconstruction segment can be selected larger to avoid the singular behavior; furthermore, because the singularity goes as the $-1/2$ power, its effect is evident only very close to the endpoints of the reconstruction segment.

C. Numerical Studies of Noise Properties in Fan-Beam Reconstruction

Using the fan-beam configuration in Fig. 1(c) and the parameters listed in Table I, we calculated fan-beam noiseless data for the numerical phantom in Fig. 1(a). We have also used an object-independent Gaussian-noise model and an object-dependent Poisson-noise model in this numerical study. The standard deviation σ_0 of Gaussian noise used is 2.3% of the maximum value in noiseless fan-beam data, whereas the standard deviation for the Poisson noise is the noiseless data scaled to yield a total photon count of 5×10^5 for each view. For each noise model, 10 000 sets of noisy data were generated by use of the corresponding noiseless data as the means. We investigated reconstruction segments of four different lengths $L_{AB} = 7.8, 10.0, 14.1,$ and 20.0 cm. All of the segments are located at $x = 4.06$ cm. It can be observed in Fig. 1(c) that the length of the support segment is 5.5 cm, which is shorter than the four reconstruction segments. Therefore, the image on this chord can be reconstructed exactly by use of data determined by these reconstruction segments. One can also conclude from Fig. 1(c) that data determined by the first three reconstruction segments, which are shorter than the maximum dimension (about 15.6 cm) of the object support, are truncated.

1) *Noise Properties in Reconstruction from Truncated Fan-Beam Data:* From the 10 000 sets of data containing Gaussian noise, we used (4)–(6) to reconstruct 10 000 noisy $\mathbf{f}_c(x_c, s_a, s_b)$, $\hat{\mathbf{f}}_c(x_c, s_a, s_b)$, and $\mathbf{g}(x_c, s_a, s_b)$, respectively, on the four reconstruction segments described. Based upon these noisy reconstructions, we subsequently computed their corresponding empirical variances, which are shown in the upper row of Fig. 8.

Fig. 6. Empirical variance images obtained by use of (a) BPF, (b) MDFBP, and (c) FBP algorithms from nontruncated parallel-beam data containing Gaussian noise (upper row) and Poisson noise (lower row). For the purpose of displaying details in central (i.e., low variance) regions, we have applied a logarithmic scale to variance images. Display windows are $[-1.3, 0.30]$ and $[-1.3, 0.40]$ for Gaussian noise and Poisson noise, respectively.

Fig. 7. Variance profiles on the dashed lines indicated in variance images shown in Fig. 6. They were obtained for the Gaussian (a) and Poisson (b) noise models by use of the BPF (solid), MDFBP (dashed), and FBP (dotted) algorithms, respectively.

As for the analytic variance, one can determine $c(x_c)$ by using $\sigma(u_d, s) = \sigma_0$ in (32), where σ_0 is 2.3% of the maximum value in noiseless fan-beam data. Using $c(x_c)$ in (23), (36), and (38), we computed analytically image variances, which are displayed in the middle row of Fig. 8. Similarly, using (4)–(6), we reconstructed 10 000 sets of noisy $\mathbf{f}_c(x_c, s_a, s_b)$, $\hat{\mathbf{f}}(x_c, s_a, s_b)$, and $\mathbf{g}(x_c, s_a, s_b)$ on the four segments from 10 000 sets of fan-beam data containing Poisson noise. The computed empirical variances from these noisy images are displayed in the upper row of Fig. 9. Furthermore, using the noiseless fan-beam data as the Poisson-noise variance $\sigma^2(u_d, s)$ in (32), one can readily determine $c(x_c)$. Using the determined $c(x_c)$ in (23), (36), and (38), we computed analytically image variances, which are displayed in the middle row of Fig. 9. It can be observed that the analytic and empirical results agree well with each other,

suggesting that (38) provides an adequate analytic estimation of the chord-image variance for the fan-beam case as well. It is interesting to note in Fig. 8(a) and Fig. 9(a) that the variances of $\mathbf{g}(x_c, s_a, s_b)$ are spatially varying on the chord. Based upon (32), one can readily conclude that this spatial variation is caused by the spatially variant factor $(A^2(u_d)/|\vec{r} - \vec{r}_0(s)|^4)$.

Again, from these results, observations similar to those for the parallel-beam case can be made for the fan-beam case. For example, as Fig. 8(c) and Fig. 9(c) show, the shorter the reconstruction segment, the higher the chord-image variances. This is only because the second term in (38) increases as L_{AB} (i.e., $(x_A - x_c)(x_c - x_B)$) decreases. The implication of these results is that there may be a significant gain in terms of dose reduction by using a short reconstruction segment, because data required to reconstruct an image on this reconstruction segment is less than that required by using a longer reconstruction segment, which can result in reduction of illumination coverage of the object.

We have also performed numerical studies of the noise properties of the reconstructed ROI images by use of the BPF and MDFBP algorithms from truncated data. Using the numerical phantom in Fig. 1 and each of the Gaussian- and Poisson-noise models described previously, we generated 500 noisy truncated data sets for image reconstruction on reconstruction segments of a length $L_{AB} = 10.0$ cm, as shown in Fig. 1(a), which cover the ROI completely. We subsequently reconstruct 500 noisy images by using each of the BPF and MDFBP algorithms. Using the reconstructed 500 sets of Gaussian-noise images and 500 sets of Poisson-noise images, we computed empirical variance images within the ROI, which are shown in the upper row and lower row of Fig. 10, respectively. Moreover, we display in the third column of Fig. 10 the variance profiles on the dashed lines

Fig. 8. Empirical (top row) and analytical (middle row) variances of (a) $g(x_c, s_a, s_b)$, (b) $\hat{f}(x_c, s_a, s_b)$, and (c) $f_c(x_c, s_a, s_b)$ obtained on four reconstruction segments from fan-beam data containing Gaussian noise. Difference between empirical variances and analytical variances is also shown in bottom row, which demonstrates the analytical variances agree well with the empirical variances. Lengths of these segments are indicated in the box in upper-right corners of plots.

indicated in the variance images. Results in Fig. 10 support the conclusion that both BPF and MDFBP algorithms yield images with comparable variance levels.

2) *Noise Properties in Reconstruction From Nontruncated Fan-Beam Data:* The FBP algorithm cannot reconstruct exactly images from truncated fan-beam data. Thus, we evaluate the noise properties of the FBP reconstruction from nontruncated fan-beam data. For the purpose of comparison, we have also included reconstruction results of the BPF and MDFBP algorithms from the same data sets. Using the numerical phantom and fan-beam configuration described above, we generated non-truncated fan-beam data at 512 views uniformly covering 2π . Using the noiseless data as the means, we generated 500 sets of data containing Gaussian noise and 500 sets of data containing Poisson noise. The standard deviation for the Gaussian noise is 2.3% of the maximum value of the noiseless data, whereas the standard deviation for the Poisson noise is the noiseless

data scaled to yield a total photon count of 5×10^5 for each view. For a given chord specified by s_a and s_b , one can reconstruct its image from data acquired over the right-side trajectory (i.e., $s \in [s_a, s_b]$), as shown in Fig. 11(a). Conversely, one can also reconstruct the chord image from data acquired with both right-side trajectory (i.e., $s \in [s_a, s_b]$) and left-side trajectory (i.e., $s \in [s_b, s_a]$), as shown in Fig. 11(b). In chord-based image reconstruction, we decompose image area into chords parallel to the vertical direction and the source scans from s_a to s_b and then from s_b to s_a , as shown in Fig. 11.

For each chord in the set covering the image area, we first reconstructed the images by use of the BPF, MDFBP, and FBP algorithms from data containing Gaussian noise acquired over the right-side trajectory specified by $s \in [s_a, s_b]$. Subsequently, we computed empirically chord-image variances from these noisy reconstructions. By assembling the chord-image variances, we obtain the variance images, which are shown in the upper row of

Fig. 9. Empirical (top row) and analytical (middle row) variances of (a) $g(x_c, s_a, s_b)$, (b) $\hat{f}(x_c, s_a, s_b)$, and (c) $f_c(x_c, s_a, s_b)$ obtained on four reconstruction segments from fan-beam data containing Poisson noise. Difference between empirical variances and analytical variances is also shown in bottom row, which demonstrates analytical variances agree well with empirical variances. Lengths of these segments are indicated in the box in upper-right corners of plots.

Fig. 12, for the BPF, MDFBP and FBP algorithms, respectively. Similarly, from data containing Poisson noise, we obtained the image variances, which are displayed in lower row of Fig. 12.

We show in column one of Fig. 13 the image variances on a chord specified by $s_a = -\pi/2$ and $s_b = \pi/2$ obtained from data containing Gaussian noise (upper row) and Poisson noise (lower row), respectively. As already seen, the variance increases with the position along the chord near the source trajectory. There is little difference between the three algorithms. Furthermore, these variance images have similar properties: the chords on the right part have higher and more nonuniform image variance than those on the left part in the image area. In column 2 of Fig. 13, we show the profiles on the middle points across the vertical chords (i.e., on the middle horizontal lines in the variance images shown in Fig. 12). The results reveal that some difference of the MDFBP result from the BPF and FBP results

in the peripheral region. This difference may be attributed to the data weighting prior to the backprojection step, which differs from that in the BPF and FBP algorithms. We should point out that this difference is only seen in the extreme periphery of the imaging area. For most practical situations these three algorithms perform virtually identically in terms of image variance.

We investigate further the general trend of the variance decreasing for chords on the left of the variance image. The first impression of this behavior is that this trend is obvious, because the scanning trajectory is on the right side, chords on the left of the variance image are reconstructed with a longer scanning trajectory. It appears that more data are used in reconstructing chords covering the left part of the variance image. This explanation is, however, incorrect. First, there is a slight upturn in the variance for the chords on the extreme left of the variance image, which runs counter to this trend. Second,

Fig. 10. Empirical ROI-image variances obtained by use of (a) BPF and (b) MDFBP algorithms from truncated fan-beam data containing Gaussian noise (upper row) and Poisson noise (lower row), respectively. (c) Variance profiles on dashed lines indicated in columns (a) and (b) obtained with BPF (solid) and MDFBP (dashed) algorithms, respectively. For the purpose of displaying details in central (i.e., low variance) regions, we have applied a logarithmic scale to variance images. Display windows are $[-0.76, 0.29]$ and $[-0.75, -0.04]$ for Gaussian noise and Poisson noise, respectively.

Fig. 11. (a) Right-side trajectory (solid) of chord (thick) specified by s_a and s_b . (b) Right-side trajectory (solid) and left-side trajectory (solid) of chord (thick) specified by s_a and s_b . Scanning configuration in (b) corresponds to a full fan-beam scan.

it can be demonstrated that the amount of data going into the chord reconstruction does not necessarily increase as the scanning trajectory increases. Based upon (32), one can conclude that the true cause of the variance behavior is spatially dependent weighting factor, $1/|\vec{r} - \vec{r}_0(\lambda)|$, in the BPF, MDFBP, and FBP algorithms [19].

For a given chord specified by s_a and s_b , when full scan data are available, one can reconstruct two chord images by use of data acquired with the right-side and left-side trajectories, as shown in Fig. 11 and then obtain a final chord image by averaging the two chord images. We show, in Fig. 14, the variance images of the full scan with accompanying profiles in Fig. 15.

D. Numerical Studies of Noise Properties in Cone-Beam Reconstruction

The BPF and MDFBP algorithms can yield exact image reconstruction on a chord specified by s_a and s_b as long as the

support segment on the chord is illuminated by the X-ray beam at the projection views $s \in [s_a, s_b]$, because these algorithms require data only on the fan-beam projections of the support segment. From the perspective of the chord-based algorithms, the reconstruction of a chord image from cone-beam data is similar to that of a chord image from fan-beam data. In the fan-beam case, the orientation of the fan-beam planes at different views remain unchanged, whereas, in the cone-beam case with a non-planar trajectory, the orientation of the fan-beam-illumination planes generally varies from view to view. As discussed in Section III-A3, the noise properties of differentiation, backprojection, and filtration in the cone-beam case are similar to that in the fan-beam case. Therefore, we include below only the study results on the noise properties of the final chord-images reconstructed from cone-beam data.

1) *Helical Cone-Beam Configuration:* In our investigation of the noise properties of image reconstruction from cone-beam data, we consider the helical trajectory, which is the most widely used in clinical and industrial CT. For a helical scan, the source trajectory is described mathematically as $\vec{r}_0(s) = (R \cos s, R \sin s, (h/2\pi)s)$, where R is the source to center-of-rotation distance, and h indicates the helical pitch. For a chord specified by s_a and s_b , if $(n-1)\pi \leq |s_b - s_a| \leq (n+1)\pi$, where n is a positive odd integer, the chord is also referred to as an $n\pi$ -line segment [20], [21], as shown in Fig. 16(a). In particular, when $n = 1$ and thus $0 \leq |s_b - s_a| \leq 2\pi$, the chord is referred to as a π -line segment [2], [11]. In this paper, we consider image reconstruction only on π -line segments for the reason that the imaging volume enclosed by the helix can be filled uniquely and completely by

Fig. 12. Empirical variance images obtained by use of (a) BPF, (b) MDFBP, and (c) FBP algorithms from nontruncated fan-beam data containing Gaussian (upper row) and Poisson (lower row) noise. For the purpose of displaying details in central (i.e., low variance) regions, we have applied a logarithmic scale to variance images. Display window is $[-1.0, 0.65]$ and $[-2.02, 0.29]$ for Gaussian noise and Poisson noise, respectively.

Fig. 13. Variance profiles along central (a) vertical and (b) horizontal lines in variance images shown in Fig. 12, obtained by BPF (solid), MDFBP (dashed), and FBP (dotted) algorithms from data containing Gaussian noise (upper row) and Poisson noise (lower row).

π -line segments [11], [12]. Thus, π -line segments can be used to form 3-D images in a helical cone-beam scan.

We computed the noiseless data from a Shepp–Logan phantom by use of the configuration parameters in Table I.

Using the noiseless data as the means, we subsequently generated 500 sets of data containing Gaussian noise and 500 sets of data containing Poisson noise, respectively. The standard deviation of Gaussian noise is chosen to be 0.7% of the maximum value of the noiseless data, whereas the standard deviation for the Poisson noise is the noiseless data scaled to yield a total count of 5×10^5 for each view.

2) *Noise Properties in Reconstruction From Helical Cone-Beam Data:* A curved surface in the helix volume can be formed by a set of π -line segments for which we fix one end-point at s_a and sweep the other endpoint over a range $s_b \in [s_{\min}, s_{\max}]$. We show in Fig. 16(a) a curved surface obtained by concatenating a set of π -line segments specified by $s_a = -\pi$ and $s_b \in [-0.5\pi, 0.5\pi]$. Using generated noisy helical cone-beam data, we reconstructed noisy images on the π -line surface by use of the BPF, MDFBP, and FBP algorithms. From these noisy images we subsequently computed empirical image variances on the π -line surface. In Fig. 17, we display the image variances obtained with BPF, MDFBP, and FBP algorithms from data containing Gaussian noise and Poisson noise.

We also display in Fig. 18(a) and (b) image variances on the π -line segment specified by $s_a = -\pi$ and $s_b = 0$ in the surface, obtained from data containing Gaussian noise and Poisson noise, respectively. The image variances show similar characteristics to that of fan-beam image variances observed in Fig. 13. Namely, the variance image on the π -line surface in Fig. 17 has a structure that is similar to the right-side scan fan-beam results presented in Section III-C-1; the images on π -line segments reconstructed from smaller helix segments tend to have higher and more nonuniform variances. The similarity with the fan-beam case is not surprising because the geometrical arrangement of

Fig. 14. Variance images obtained by use of (a) BPF, (b) MDFBP, and (c) FBP algorithms from full-scan fan-beam data containing Gaussian (upper row) and Poisson (lower row) noise. For the purpose of displaying details in central (i.e., low variance) regions, we have applied a logarithmic scale to variance images. Display windows are $[-1.19, 0.31]$ and $[-1.18, 0.20]$ for Gaussian noise and Poisson noise, respectively.

Fig. 15. Variance profiles along the central (a) vertical and (b) horizontal lines in variance images, which are shown in Fig. 14, obtained by BPF (solid), MDFBP (dashed), and FBP (dotted) algorithms from data containing Gaussian noise (upper row) and Poisson noise (lower row).

the π -line with respect to its scanning trajectory is very similar to the relationship between the chords and corresponding fan-beam scanning trajectory. The only difference is that there is an out-of-plane bent to the helix segment.

Fig. 16. (a) π -line, 3π -line, and 5π -line segments in a helical scan. (b) Surfaces generated in imaging volume by concatenating π -line segments specified by $s_a = -\pi$ and $s_b \in [-0.5\pi, 0.5\pi]$.

Regarding the nonuniform shape of the variance, one can attribute the high variance in the image periphery to the weighting factors multiplying the data derivatives before backprojection. As the algorithms are essentially the same for chord-image reconstruction in fan- and cone-beam cases, this conclusion should come as no surprise. In the 2-D fan-beam case, the variance nonuniformity and level was reduced by equally weighting reconstructions for both left and right side scans for each chord of the scanning circle. For the helical configuration, it is clear that in a typical scan there will be some overscan for nearly all the chords comprising the volume. But the overscan part of the trajectory does not form a closed loop so using the overscan data to reduce image variance is not as obvious as the case of the circular scan. Future work will focus on how to utilize the overscan data for non-closed trajectories for the purpose of reducing the impact of data noise on chord-base ROI-image reconstruction.

Fig. 17. Empirical variance images on π -line surface shown in Fig. 16(b) obtained by use of (a) BPF, (b) MDFBP, and (c) FBP algorithms from data containing Gaussian noise (upper row) and Poisson noise (lower row), respectively. For the purpose of displaying details in central (i.e., low variance) regions, we have applied a logarithmic scale to variance images. Display windows are $[-1.80, 0.30]$ and $[-1.80, 0.60]$ for Gaussian noise and Poisson noise, respectively.

Fig. 18. Variances on central vertical line segment, specified by $s_a = -\pi$ and $s_b = 0$, obtained with BPF (solid), MDFBP (dashed), and FBP (dotted) algorithms from data containing (a) Gaussian noise and (b) Poisson noise.

IV. DISCUSSION

In this paper, we have performed analytic and numerical investigations of the noise properties of chord-based image reconstructions from parallel-, fan-, and cone-beam data. One of the main points of the investigation was to test whether or not the reduced illumination in designing a minimal data set for a particular ROI leads to a significant reduction in exposure. The idea was to compare the statistical properties of the ROI image reconstructed from noise realizations of the minimal data set with noise realizations of the full data set. Similar noise levels were used in both data sets, which are equivalent to modeling similar incident X-ray beam intensities. Our study indicates that the resulting image variance was almost the same for images reconstructed from both data sets.

Thus, the minimal data set for ROI reconstruction leads to a significant overall dose reduction, because the body is exposed to lower amount of ionizing radiation in the reduced scan. For fan-beam and cone-beam imaging, we explored the noise properties of the extreme periphery of the imaging region by investigating large fan- and cone-angles. Image variance nonuniformity was found to be caused by spatially dependent weighting factors in the chord-based reconstruction algorithms. This work represents a study of the noise properties of chord-based reconstruction and of the impact of physical factors on ROI imaging in fan-beam and cone-beam CT. In seeking ways to reduce the impact of noise in volume imaging, we will investigate schemes to incorporate overscan data. The analysis presented in this work can directly be applied to chord-based image reconstruction for general trajectories. Finally, it is important to investigate the behavior of the ROI-reconstruction algorithms when other important factors are included in the data model such as X-ray polychromaticity and nonlinear partial volume averaging.

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