A Sufficient Condition for Uniform Convergence of Stationary *p*-Subdivision Scheme

Yi-Kuan Zhang[1, 2], Ke Lu[3], Jiangshe Zhang[1], and Xiaopeng Zhang[2,*]

[1] School of Science, Xi'an Jiaotong University, Xi'an 710049, China
[2] LIAMA-NLPR (National Laboratory of Pattern Recognition), Institute of Automation, Chinese Academic of Science, Beijing 100080, China
[3] Graduate University of Chinese Academy of Science, Beijing, 100049 China
ykzhang@liama.ia.ac.cn, luk@gscas.ac.cn,
jszhang@mail.xjtu.edu.cn, xpzhang@nlpr.ia.ac.cn

Abstract. Subdivision is a convenient tool to construct objective curves and surfaces directly from given scattered points. Stationary *p*-subdivision schemes are highly efficient in the acquisitions of curve/surface points in shape modeling. The features of supported set of nonnegative mask of uniform convergent stationary subdivision schemes are important to their theoretic researches and applications. According to the properties of supported set of the nonnegative mask, a sufficient condition for uniform convergence of stationary *p*-subdivision scheme is presented. This condition is proved with two propositions and spline function. The contribution of this work is that the convergence of a stationary *p*-subdivision scheme can be judged directly. This direct judge is in favor of applications of this scheme.

Keywords: geometric modeling, stationary *p*-subdivision, uniform convergence, contractility, spline function.

1 Introduction

Stationary subdivision schemes arise from modeling and interrogation of curves and surfaces, image decomposition and reconstruction, and the problems of constructing compact supported wavelet basis etc. [1, 9]. These schemes are being developed in geometric modeling with great potentiality in CAD/CAM, computer graphics, image processing, etc. [1-11, 14, 15]. Stationary subdivision schemes are widely used in mechanical CAD, garment CAD, jewellery CAD, and applied in computer graphics. They also play important roles in image coding, signal processing, and the construction of basis function of compact supported orthogonal wavelets by using multiresolution analysis [1-3, 17-23]. They are also important in fractal and its generation by computer in particularly [1, 2, 4, 12]. Stationary subdivision schemes are used to construct the required curves and surfaces from scattered data directly through stated subdivision rules. Moreover the theoretical contribution of this approach consists in their tight combination of three research disciplines: spline functions, wavelets and

[*] Corresponding author.

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fractals [1-4, 9, 10, 12]. Therefore, the research of stationary subdivision schemes, especially its convergence, is significant in theoretical research and shape modeling [2, 3, 5-13]. The idea and approaches of stationary subdivision schemes are still effective in subdivision surfaces [24, 25,26] and the constructions of compactly supported orthogonal wavelets basis and fractal [11, 12, 13, 16, 24].

The systematic development of the basic mathematical principles and concepts associated with stationary 2-subdivision schemes is presented in [1]. The structure of these algorithms in a multidimensional setting and convergence issue are researched systematically. The complete theoretical system is constructed. The analytic structure of limit curves and surfaces generated by these algorithms is revealed [1, 16].

The extension of stationary 2-subdivision to stationary p-subdivision scheme is presented in [9]. Some properties of convergence of such schemes are described through Fourier analysis, functional analysis and spline function. A sufficient condition of the uniform convergence of the stationary p-subdivision scheme is discovered in [10] through a special polygon, δ-control polygon.

The problem is important of how to use this kind of subdivision schemes to generate curves and surfaces in computer graphics [1-8, 18-21]. The convergence of stationary subdivision schemes is a key problem in the theory of stationary subdivision scheme and their applications [1, 9, 10]. Finding the features of the support set of nonnegative mask $\mathbf{a} = \{a_\alpha : \alpha \in \mathbf{Z}^s\}$ [1, 9, 10] has an important value in theoretical researches and practical applications, because the convergence of these algorithms can be judged directly in the construction of curves and surface. So, the sufficient conditions of the uniform convergence of stationary p-subdivision schemes based on the supported set of mask may promote theoretical researches and practical applications [1, 9, 10].

A sufficient condition of the uniform convergence of stationary p-subdivision schemes is presented in this paper using contractility and spline function. This work is based on three aspects: the nonnegative mask and its support set of stationary p-subdivision schemes, some definitions and properties of stationary p-subdivision schemes presented in [9, 10], and the works of [1-5].

Here is the main theorem of this paper.

Theorem. The stationary p-subdivision scheme

$$\lambda^0 = \lambda, \quad \lambda^m = S\lambda^{m-1}, \qquad m = 1, 2, \cdots$$

defined in (2) is uniformly convergent, if the positive mask $\mathbf{a} = \{a_\alpha : \alpha \in \mathbf{Z}^s\}$ supported on Ω satisfies

$$\sum_{\beta \in \mathbf{Z}^s} a_{\alpha - p\beta} = 1, \quad \alpha \in \mathbf{Z}^s,$$

where

$$\Omega := \mathbf{Z}(A) \bigcap \mathbf{Z}^s \text{ , zonotope } \mathbf{Z}(A) := \{Au : u \in \prod_{i=1}^{s} [l_i, u_i]\}, \ l_i + p - 1 < u_i, i = 1, \cdots, s .$$

2 Preliminaries and Propositions

Six definitions and two propositions are introduced in order to prove of above theorem.

Definition 1. Let s be a fixed natural number and $\mathbf{Z}^s$ the integer lattice, and $\mathbf{a}=\{a_\alpha:\alpha\in\mathbf{Z}^s\}$ be the fixed real scalar sequences having finitely supported set supp**a**$=\{\alpha:a_\alpha\neq 0\}$. A stationary p-subdivision operator S is defined as

$$S:\ell^\infty(\mathbf{Z}^s)\to\ell^\infty(\mathbf{Z}^s) \tag{1}$$

by

$$(S\lambda)_\alpha=\sum_{\beta\in\mathbf{Z}^s}a_{\alpha-p\beta}\lambda_\beta\,,\quad \lambda\in\ell^\infty(\mathbf{Z}^s)\,,$$

where $p>1$ is a fixed natural number, and λ is point sequence.

Definition 2. Let S be any stationary p-subdivision operator defined in (1), the following iteration scheme

$$\lambda^0=\lambda,\quad \lambda^m=S\lambda^{m-1},\qquad m=1,2,\cdots \tag{2}$$

is defined as a stationary p-subdivision scheme. $\mathbf{a}=\{a_\alpha:\alpha\in\mathbf{Z}^s\}$ is referred to as the mask of the stationary p-subdivision scheme S. λ is called as the control polygon of S. In fact, λ_α is a vertex of the control polygon λ.

Definition 3. The stationary p-subdivision scheme (2) is said to be convergent for $\lambda\in\ell^\infty(\mathbf{Z}^s)$ if there exists a continuous function $f_\lambda\in\mathbf{C}^0(\mathbf{R}^s)$, such that

$$\lim_{m\to\infty}\left\|f_\lambda\left(\frac{\bullet}{p^m}\right)-\lambda^m\right\|_\infty=0\,. \tag{3}$$

Definition 4. The p-subdivision scheme (2) is said to be uniformly convergent if there exists a continuous function $f_\lambda\in\mathbf{C}^0(\mathbf{R}^s)$ for all $\lambda\in\ell^\infty(\mathbf{Z}^s)$, such that

$$\lim_{m\to\infty}\left\|f_\lambda\left(\frac{\bullet}{p^m}\right)-\lambda^m\right\|_\infty=0\,. \tag{4}$$

Stationary p-subdivision algorithms (1) actually have p^s different subdivision rules, and the norm is defined as $\|\lambda\|_\infty=\sup_{\alpha\in\mathbf{Z}^s}|\lambda_\alpha|$ in $\ell^\infty(\mathbf{Z}^s)$ in the above definitions. The control polygon λ is represented as scalar-valued, i.e. $\lambda\in\ell^\infty(\mathbf{Z}^s)$, in this paper, since the influence on λ of stationary p-subdivision scheme S in (1) and (2) is performed as that of coordinate components of vertices.

The basic difference of stationary p-subdivision schemes and stationary 2-subdivision schemes is that stationary p-subdivision schemes have p^s different rules while stationary 2-subdivision schemes have 2^s different rules. If $p>2$ and the two kind of stationary subdivision schemes in (2) are convergent, stationary p-subdivision schemes can be used to generate curves or surfaces by with fewer iterative steps than stationary 2-subdivision schemes, so stationary p- subdivision schemes are subdivision schemes having a faster convergence speed and a higher efficiency in curve/surface modeling.

Stationary p-subdivision schemes and their some basic convergent properties are presented in [9]. A sufficient condition for uniform convergence of stationary p-subdivision schemes is given in [10] by using a special control polygon of δ-control polygon.

$D:\ell^\infty(\mathbf{Z}^s)\to\mathbf{R}^+\bigcup\{0\}$ is thought as a no-trivial non-negative functional in the following description.

Definition 5. A stationary p-subdivision operator S is said to be contractive relative to D if there exists a constant number γ ($0<\gamma<1$) for the subdivision operator S defined by (1), such that

$$D(S\lambda)\le\gamma D(\lambda)\,,\quad \lambda\in\ell^\infty(\mathbf{Z}^s)\,. \tag{5}$$

Suppose $\mu\in\mathbf{R}^s$ is a fixed vector not necessarily a lattice point, and $\sup p\mathbf{a}\subseteq\Omega:=(\mu+\mathbf{\Gamma})\bigcap\mathbf{Z}^s$, where $\mathbf{\Gamma}\subseteq\mathbf{R}^s$ be a balanced convex closed set which corresponding Minkowski functional is ρ, then $y\in\mathbf{\Gamma}$ if and only if $\rho(y)\le 1$.

Definition 6. For any control polygon $\lambda\in\ell^\infty(\mathbf{Z}^s)$,

$$D_\rho(\lambda):=\sup_{\substack{\rho(\alpha-\beta)<2\\ \alpha,\beta\in\mathbf{Z}^s}}\left|\lambda_\alpha-\lambda_\beta\right| \tag{6}$$

is defined as a diameter of λ.

The convergent condition of stationary p-subdivision scheme will be discussed in the following under the condition of that the support of mask $\mathbf{a}=\{a_\alpha:\alpha\in\mathbf{Z}^s\}$ is the union of $\mathbf{Z}^s$ and a special zonotope $\mathbf{Z}(A):=\{Au:u\in\prod_{i=1}^{s}[l_i,u_i]\}$, where A is a $s\times s$ integer matrix, and $\det A=-1$. The following two propositions are used to prove the main theorem.

Proposition 1. Let $\mathbf{a}=\{a_\alpha:\alpha\in\mathbf{Z}^s\}$ be any mask satisfying following conditions:

$$a_\alpha\ne 0\,,\ \textit{implies}\quad \alpha\in\Omega\,, \tag{7}$$

$$\sum_{\beta\in\mathbf{Z}^s}a_{\alpha-p\beta}=1,\quad \textit{for}\ \forall\alpha\in\mathbf{Z}^s\,; \tag{8}$$

and $D_\rho(\lambda)$ be functional defined by (6) on $\ell^\infty(\mathbf{Z}^s)$. Then stationary p-subdivision operator S defined by (1) satisfies:

$$D(S\lambda)\le\gamma_\rho D_\rho(\lambda)\ , \tag{9}$$

Where

$$\gamma_\rho=\frac{1}{2}\max_{\rho(\sigma-\delta)<2}\sum_{\beta\in\mathbf{Z}^s}\left|a_{\sigma-p\beta}-a_{\delta-p\beta}\right|. \tag{10}$$

Lack of space forbids the proof of this proposition here.

Proposition 2. Under the following three conditions

(i) B be a p-subdivision operator which has finitely supported mask $\mathbf{b} = \{b_\alpha : \alpha \in \mathbf{Z}^s\}$ and stable refinable function ψ, and the corresponding stationary p-subdivision scheme is uniform convergent: $(B\lambda)_\alpha := \sum_{\beta \in \mathbf{Z}^s} b_{\alpha - p\beta} \lambda_\beta$. Where ψ is a stable refinable function means that for refinable function ψ there exists a positive constant $C_1 > 0$ such that

$$C_1 \|\lambda\|_\infty \le \left\| \sum_{\alpha \in \mathbf{Z}^s} \lambda_\alpha \psi(\bullet - \alpha) \right\|_\infty \le C_{21} \|\lambda\|_\infty , \tag{11}$$

Where $C_2 := \left\| \sum_{\alpha \in \mathbf{Z}^s} |\psi(\bullet - \alpha)| \right\|_\infty$.

(ii) Stationary p-subdivision operator S defined by (1) is contractive relative to functional D.

(iii) There exists a constant C, such that

$$\|S\lambda - B\lambda\|_\infty \le C \cdot D(\lambda), \quad \lambda \in \ell^\infty(\mathbf{Z}^s). \tag{12}$$

The following two conclusions can be obtained

(a) The stationary p-subdivision scheme determined by S is uniformly convergent.

(b) If the condition (11) is replaced by the following condition that there exist a constant C such that

$$|\lambda_\alpha - \lambda_\beta| \le C \cdot D(\lambda), \quad \beta - \alpha \in (\sup p\psi)^\circ, \tag{13}$$

the stationary p-subdivision scheme determined by S is also uniformly convergent.

Proof: For the definitions in (1) and (2), we can obtain that [10]

$$\lambda_\alpha^m = (S^m \lambda)_\alpha = \sum_{\beta \in \mathbf{Z}^s} a^m_{\alpha - p^m \beta} \lambda_\beta \text{ and } a^m_\alpha = (Sa^{m-1})_\alpha = \sum_{\beta \in \mathbf{Z}^s} a^{m-1}_\beta a_{\alpha - p\beta}. \tag{14}$$

If define f_λ^m as follows

$$f_\lambda^m(x) = \sum_{\alpha \in \mathbf{Z}^s} \lambda_\alpha^m \psi(p^m x - \alpha), m = 1, 2, \cdots,$$

the conclusion of proposition 2 can be proved with following inequality [10]

$$\left\| f_\lambda(\frac{\bullet}{p^m}) - \lambda^m \right\| \le \left\| f_\lambda(\frac{\bullet}{p^m}) - f_\lambda^m(\bullet) \right\| + \left\| f_\lambda^m(\bullet) - \lambda^m \right\|.$$

The detailed proof of this proposition is omitted here.

3 The Proof of the Main Theorem

Now we give the proof of theorem presented in this paper on the base of the proposition 1 and proposition 2. In the proof, above definition of contractility and spline function will be used.

Proof: (i) Let $\Omega := \mathbf{Z}(A) \bigcap \mathbf{Z}^s$, then it is a hyperrectangle according to the definition of $\mathbf{Z}(A)$. So we can suppose that $\mathbf{Z}(A) := \{Au : u \in \prod_{i=1}^{s} [l_i^1, u_i^1]\}$. Then according to the hypothesis of that Ω is the support of mask $\mathbf{a}$, we know that:

$$a_{\sigma-p\beta}, a_{\delta-p\beta} > 0 \quad \Leftrightarrow \quad l_i^1 \le \sigma_i - p\beta_i, \delta_i - p\beta_i \le u_i^1, i = 1, 2, \cdots, s \tag{15}$$

So, if let

$$\mu = \frac{1}{2}(u^1 - l^1),\ u^1 = (u_1^1, u_2^1, \cdots, u_s^1),\ l^1 = (l_1^1, l_2^1, \cdots, l_s^1),$$

$$\rho(x) := 2 \max_{1 \le i \le s} \left| \frac{\sigma_i - \delta_i}{u_i^1 - l_i^1} \right|,\ x = (x_1, x_2, \cdots, x_s),$$

and $\mathbf{\Gamma}$ is the set determined by Minkowski functional $\rho(x)$, as a result , we know that $\Omega = (\mu + \mathbf{\Gamma}) \bigcap \mathbf{Z}^s$ from the definition of μ and $\mathbf{\Gamma}$, so $a_\alpha > 0$ if and only if $\alpha \in \Omega = (\mu + \mathbf{\Gamma}) \bigcap \mathbf{Z}^s$.

Moreover, the mask $\mathbf{a} = \{a_\alpha : \alpha \in \mathbf{Z}^s\}$ satisfies (8) because of known conditions. So, for $D_\rho(S\lambda)$ defined with (6), we conclude that $D_\rho(S\lambda) \le \gamma_\rho D(\lambda)$ according to proposition 1. Thus from that a_α $(\alpha \in \mathbf{Z}^s)$ is positive on Ω, it follows that

$$\gamma_\rho = \frac{1}{2} \max_{\rho(\sigma-\delta)<2} \sum_{\beta \in \mathbf{Z}^s} \left| a_{\sigma-p\beta} - a_{\delta-p\beta} \right| \le \frac{1}{2} \max_{\rho(\sigma-\delta)<2} \Big(\sum_{\beta \in \mathbf{Z}^s} a_{\sigma-p\beta} + \sum_{\beta \in \mathbf{Z}^s} a_{\delta-p\beta} \Big) = 1$$

Therefore, if find a β satisfying (15) whenever $\left|\sigma_1 - \delta_1\right| < u_i^1 - l_i^1 = u_i - l_i$. Then for such β $\left| a_{\sigma-p\beta} - a_{\delta-p\beta} \right| < a_{\sigma-p\beta} + a_{\delta-p\beta}$ is true. Thus $\gamma_\rho < 1$.

(ii) To determine the β satisfying above requirement in the following.

From the inequalities in (15) above, it may be known that in order to make the $a_{\sigma-p\beta} > 0,\ a_{\delta-p\beta} > 0$ true the subscripts $\sigma - p\beta,\ \delta - p\beta$ should satisfy:

$$l^1 \le \sigma - p\beta \le u^1, l^1 \le \delta - p\beta \le u^1 . \tag{16}$$

So the expression $\left|\sigma - \delta\right| < \left|u^1 - l^1\right|$ is true since $\left|\sigma_i - \delta_i\right| < \left|u_i^1 - l_i^1\right|,\ i = 1, 2, \cdots, s$. So

$$\rho(\sigma - \delta) := 2 \max_{1 \le i \le s} \left| \frac{\sigma_i - \delta_i}{u_i^1 - l_i^1} \right| < 2 .$$

Without loss of generality, let $\sigma > \delta$, then $0 \le \sigma - \delta < u^1 - l^1$, and so $\sigma - u^1 < \delta - l^1$. Now to solve the $p\beta$ from the expression (16), the result $\sigma - u^1 \le p\beta \le \delta - l^1$ is obtained. Therefore there always exists a integer in interval $[\sigma - u^1, \delta - l^1]$, so that this integer is $p\beta$, so the β can be find out according to the each coordinate component, and such $\beta \in \mathbf{Z}^s$ is satisfies all the inequalities (15). Therefore the operator S has following property known from the conclusion (i):

$$D(S\lambda) \le \gamma_\rho D_\rho(\lambda) \text{ , and } 0 < \gamma_\rho < 1 .$$

(iii) To construct an operator B defined in (1), which has finite support $\mathbf{b}=\{b_\alpha:\alpha\in\mathbf{Z}^s\}$ and refinable function ψ, and the B makes the corresponding stationary p-subdivision scheme be uniform convergent and satisfying (13).

Firstly, let $\varphi_1(t)$ is a B-spline function of degree one: $\varphi_1(t)=\begin{cases}1-|t| & ,|t|\le 1\\ 0 & ,others\end{cases}$. Now let $\varphi(x)=\prod_{i=1}^{s}\varphi_1(x_i)$, $x=(x_1,x_2,\cdots,x_s)$, if let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)$, then

$$\sum_{\alpha\in\mathbf{Z}^s}\varphi(x-\alpha)=\sum_{\alpha_1,\alpha_2,\cdots,\alpha_s\in\mathbf{Z}^s}\varphi_1(x_1-\alpha_1)\varphi_1(x_2-\alpha_2)\cdots\varphi_1(x_s-\alpha_s)=\prod_{i=1}^{s}\sum_{\alpha\in\mathbf{Z}^s}\varphi_1(x_i-\alpha_i)$$

Considering $\sum_{\alpha\in\mathbf{Z}^s}\varphi_1(x_i-\alpha_i)=1$, so $\sum_{\alpha\in\mathbf{Z}^s}\varphi(x-\alpha)=1 \quad x\in\mathbf{R}^s$ (17)

Moreover if we let

$$b_\alpha=b_{\alpha_1}b_{\alpha_2}\cdots b_{\alpha_s},\ \alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s),\ b_j=\begin{cases}1-1/p|j| & ,\quad j\in\{-1,0,1\}\\ 0 & ,\quad others\end{cases},$$

then the φ satisfies the p-scale equation: $\varphi(x)=\sum_{\alpha\in\mathbf{Z}^s}b_\alpha\varphi(px-\alpha) \quad ,x\in\mathbf{R}^s$. (18)

Now select $\eta=(\eta_1,\eta_2,\cdots,\eta_s)$, such that $l_i<\eta_i<u_i \quad ,i=1,2,\cdots,s$, and let $\psi(x)=\varphi(x-\eta)$, then $\mathrm{supp}\,p\mathbf{b}\subseteq\mathrm{supp}\,p\mathbf{a}$, and mask $\mathbf{b}$ is associated to ψ.

Since $\sum_{\beta\in\mathbf{Z}^s}a_{\alpha-p\beta}=1=\sum_{\beta\in\mathbf{Z}^s}b_{\alpha-p\beta}$, *for* $\alpha\in\mathbf{Z}^s$, then there is following conclusion:

$$(S\lambda)_\alpha-(B\lambda)_\alpha=\Big(\sum_{\beta\in\mathbf{Z}^s}a_{\alpha-p\beta}\lambda_\beta-\sum_{\beta\in\mathbf{Z}^s}Ca_{\alpha-p\beta}\Big)+\Big(\sum_{\beta\in\mathbf{Z}^s}Cb_{\alpha-p\beta}-\sum_{\beta\in\mathbf{Z}^s}b_{\alpha-p\beta}\lambda_\beta\Big),$$

where C is an arbitrary constant, B is determined by mask $\mathbf{b}=\{b_\alpha:\alpha\in\mathbf{Z}^s\}$.

For $\forall\beta\in\mathbf{Z}^s$, we can chosen proper C, to make $|\lambda_\beta-C|\le \frac{1}{2}D(\lambda)$ true. So,

$$\left|(S\lambda)_\alpha-(B\lambda)_\alpha\right|=\left|\sum_{\beta\in\mathbf{Z}^s}(\lambda_\beta-C)(a_{\alpha-p\beta}-b_{\alpha-p\beta})\right|\le D(\lambda) \quad , \quad \alpha\in\mathbf{Z}^s. \qquad (19)$$

The hypothesis conditions in proposition 2 are all true as shown in expressions (17), (18), and (19). Thus the theorem is true.

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